Revision, Defeasible conditionals and Non-Monotonic Inference for Abstract Dialectical Frameworks

Jesse Heyninck[1,2,3,4*] Gabriele Kern-Isberner[1] Tjitze Rienstra[5] Kenneth Skiba[6]
Matthias Thimm[6]
[1]Technische Universität Dortmund, Dortmund, Germany
[2]University of Cape Town and CAIR, South-Africa
[3]Vrije Universiteit Brussel, Belgium
[4]Open Universiteit, the Netherlands
[5]Maastricht University, The Netherlands
[6]FernUniversität in Hagen, Germany

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Abstract

For propositional beliefs, there are well-established connections between belief revision, defeasible conditionals, and nonmonotonic inference. In argumentative contexts, such connections have not yet been investigated. On the one hand, the exact relationship between formal argumentation and nonmonotonic inference relations is a research topic that keeps on eluding researchers despite recently intensified efforts, whereas argumentative revision has been studied in numerous works during recent years. In this paper, we show that relationships between belief revision, defeasible conditionals, and nonmonotonic inference similar to those in propositional logic hold in argumentative contexts as well. We first define revision operators for abstract dialectical frameworks, and use such revision operators to define dynamic conditionals by means of the Ramsey test. We show that such conditionals can be equivalently defined using a total preorder over three-valued interpretations, and study the inferential behaviour of the resulting conditional inference relations.

1 Introduction

In propositional logic, there are strong connections between belief revision, defeasible conditionals, and nonmonotonic inference. These connections proved to be crucial to understand common basic semantic structures underlying reasoning in all three areas, and also helpful when transferring techniques and results from one area to another. Such connections have not yet been investigated in an argumentative setting. The goal of this paper is to take first steps of generalising the connections between belief revision, defeasible conditionals and nonmonotonic inference to argumentation frameworks and present first insights from that, in particular, regarding the role that conditionals can play to connect reasoning and revision in argumentation frameworks. We choose abstract dialectical frameworks as a general argumentation formalism that can subsume many approaches to argumentation. In the following, we introduce into the fields of conditionals, nonmonotonic inference, belief revision, and formal argumentation in a bit more detail, and motivate our cross-fields approach by indicating gaps in state-of-the-art research works.

Belief revision, defeasible conditionals and nonmonotonic inference relations form a triangle of strongly connected concepts within knowledge representation. *Conditionals* [53] have been a cause of concern for philosophers for the better part of the history of philosophy, but within the formal logical study of conditionals, in the last half century, a lot of progress has been made. A central idea in the study of conditionals is that in the evaluation of a conditional "if ϕ then ψ" (formally, $(\psi|\phi)$), it suffices to check for the validity of ψ in a certain subset of all models

*Corresponding author. E-mail: jesse.heyninck@ou.nl. Adress: Jesse Heyninck, Open Universiteit, Valkenburgerweg 177, 6419 AT Heerlen, Nederland

of ϕ. This is often modelled using a selection function over the set of possible worlds Ω: $f : \Omega \times \wp(\Omega) \to \wp(\Omega)$. A conditional $(\psi|\phi)$ is then true in a world ω according to a selection function f iff every world in $f(\omega, [\phi])$ validates ψ. *Nonmonotonic inference relations* [43, 58], on the other hand, have been studied semantically using a *preference relation* $\preceq$ over the set of possible worlds. A nonmonotonic inference $\phi \mathrel{|\!\sim} \psi$ is then valid iff ψ holds in all $\preceq$-minimal ϕ-worlds. The relations between conditionals and nonmonotonic inference relations are clear, then, as $\min_{\preceq}$ can be viewed as a selection function. As such, a conditional inference relation $\mathrel{|\!\sim}$ can be associated with a nonmotonic inference relation s.t. $\preceq \mathrel{|\!\sim} (\psi|\phi)$ iff $\phi \mathrel{|\!\sim} \psi$. *Belief revision* studies the effect of the dynamics of propositional beliefs, and the consolidation of belief revision as a field of study is often identified with the formulation of the *AGM-theory* [2] of belief revision. Close relationships between belief revision and conditional logics were noticed by means of the *Ramsey test* [56], which says that a conditional $(\psi|\phi)$ is valid if ψ is believed after revision with the antecedent ϕ. The Ramsey test also gave rise to impossibility results on the compatibility of belief revision and conditional reasoning [28]. However, when [40] showed that total preorders underlie AGM-belief revision in a fundamental and inevitable way, it was at once also established that belief revision, conditional logic, and nonmonotonic inference were shown to be fully compatible. They can thus be seen as three different sides of a single topic or mode of reasoning [29, 50], at least when restricted to propositional beliefs (cf. Figure 2). Indeed, when moving to other kinds of belief revision (e. g. [31, 17]), weaker kinds of conditionals [32, 51] or other forms of nonmonotonic inference, these interrelations tend to break down or are not investigated. For example, for revision in Horn-theories, [17] has shown that rational revision operators cannot be straightforwardly represented in terms of total preorders, thus severing the link between belief revision and nonmonotonic inference. It was shown that for revision operators in Horn theories satisfying additional postulates, semantics in terms of total preorders are sound and complete, but no investigations in corresponding non-monotonic inference relations have been made. For revision of logic programs under the answer set semantics, similar complications in the characterisation of revision operators in terms of total preorders were discovered [16]. For other formalisms, such as abstract argumentation [23], revision has been widely studied [11, 20, 46], but no correspondence to nonmonotonic inference or conditional logic has been shown. This is perhaps partially due to the fact that most work on revision in argumentation has been in terms of *extension-based semantics*, for which inference in terms of complex propositional formulae is not straightforwardly defined.

Formal argumentation is an important field in knowledge representation. The true nature of the relationship between nonmonotonic and argumentative reasoning, however, is not yet fully understood. Indeed, argumentation and nonmonotonic reasoning are perceived as two different fields that do not subsume each other, and often, attempts to transform reasoning systems from one side into systems of the other side have been revealing gaps that could not be closed (cf., e. g., [63, 42, 33]). While one might argue that this is due to the seemingly richer, dialectical structure of argumentation, in the end the evaluation of arguments often boils down to comparing arguments with their attackers, and comparing degrees of belief is a basic operation in qualitative nonmonotonic reasoning. Therefore, in spite of the abundance of existing work studying connections between the two fields, the true nature of the relationship between argumentation and nonmonotonic reasoning has not been fully understood. On the other hand, belief dynamics in general and belief revision in particular has been studied intensively for formal argumentation. Therefore, in this paper we make a systematic and general attempt to answer the question as to whether belief revision, nonmonotonic inference relations, and defeasible conditionals form an interconnected triangle in an argumentative context as well. We answer this question for *Abstract Dialectical Frameworks* (ADFs) [8], an approach to formal argumentation, which subsumes other argumentative formalisms, such as set-based argumentation frameworks [24, 1, 54], argumentation frameworks with recursive attacks [54], and bipolar argumentation frameworks [54] in a generic, logic-based way. This means that any technique developed for ADFs is immediately available to the rich variety of frameworks representable in ADFs.

We illustrate this with an informal example:

Example 1. Making travel plans based in Germany, there are three candidate destinations: Addis Aba (Ethiopia), Boston (USA), and Cochem (Germany). There is not enough time to make two intercontinental travels, but when making at most one intercontinental travel, you will have enough money and time for an additional holiday in Germany. When you would make two intercontinental travels, no time for traveling to Cochem would be left.

Argumentation can be used to make an informed decision in this scenario: there are three arguments a, b and c for the three respective destinations. a and b attack each other, whereas $\{a, b\}$ attack c. We have represented this as an ADF consisting of three arguments a, b and c with their respective *acceptance conditions* C_a, C_b and C_c, whose intuitive meaning is clear by the above description and whose formal meaning will be explained in the next section, in Figure 1. Semantics like the preferred semantics (defined and explained below) allow us to

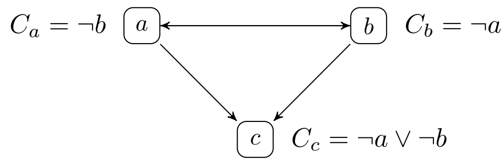

Figure 1: Argumentative representation of Example 1.

derive that both $\{a, c\}$ and $\{b, c\}$ are acceptable positions in this argumentative context: i. e. we can make one intercontinental travel and one travel in Germany.

The argumentative formalisation does not tell us, however, how we should adapt our beliefs in view of changing information. For example, suppose that a highly infectious disease breaks out in Cochem. In that case, argumentative semantics do not give information about what can be expected, unless we change the ADF in view of this information and recalculate the semantic interpretations for this new ADF. However, it might be useful to have an indication of what can be expected in the face of dynamic information. For example, is it reasonable to expect we can still make an intercontinental travel when we do not travel to Cochem (i.e. $\neg c \mathrel{|\!\sim} a \vee b$)? The derivation of such statements about what can be expected requires the investigation of belief revision and the resulting dynamic conditionals in the setting of formal argumentation.

In this paper, we investigate connections between belief revision, nonmonotonic inference, and defeasible conditionals within abstract dialectical argumentation. We first define and study revision of ADFs in depth and then use these revisions to define conditional inference for ADFs. Then, we define dynamic nonmonotonic inference relations based on the *Ramsey test* [56]. We study these inference relations in terms of rationality postulates known from defeasible conditionals. The contributions of this paper are therefore the following:

- A thorough study of revision operators for abstract dialectical argumentation under two- and three-valued semantics, including the definition of revision operators for ADFs and the semantical characterisation of such operators in terms of total preorders over the two-valued, respectively three-valued, interpretations. This study also includes an alternative development of revision operators based on strong equivalence instead of equivalence, and a problematisation of revision under possibilistic logic.
- The definition of *dynamic conditional inference relations* for ADFs based on the Ramsey test.
- A study of dynamic conditional inference relations in terms of postulates known from defeasible conditionals.

These contributions have the following broader impact:

- As ADFs subsume many other argumentative formalisms, such as set-based argumentation frameworks [24, 1, 54], argumentation frameworks with recursive attacks [54], and bipolar argumentation framework [54], the concepts developed in this paper and the corresponding results immediately apply to a wide variety of argumentative formalisms.
- By connecting the fields of argumentation and conditional inference, we bridge two fields that have hitherto existed without much interaction, show where the differences between the two approaches lie, and how these formalisms can complement each other.

Outline of this Paper: We first state all the necessary preliminaries in Section 2 on propositional logic (Section 2.1), three-valued logic (Section 2.2), reasoning with nonmonotonic conditionals (Section 2.3), propositional revision (Section 2.4), and abstract dialectical argumentation (Section 2.5). We then define revision of ADFs. We first make some general remarks on the type of revision we consider in Section 3, and then define and characterise revision under the two-valued model semantics (Section 4) and the stable semantics (Section 5). We thereafter turn to three-valued semantics (Section 6), first showing the impossibility of a revision operator under admissible and complete semantics in Section 6.2 and then defining and characterising revision under the preferred semantics (Section 6.3) and the grounded semantics (Section 6.4). In Section 6.5, we show that revision of ADFs cannot be straightforwardly modelled by revision of the acceptance conditions of an ADF. In Section 7 we define and characterise further revision operators, based on strong instead of classical equivalence. In Section 8, we

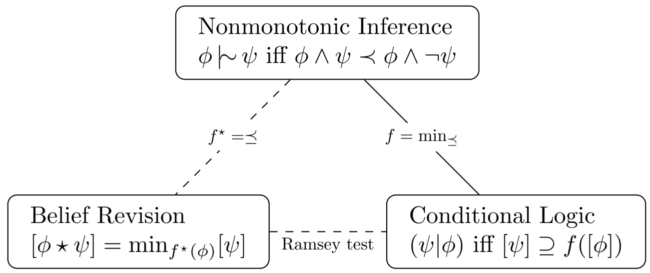

Figure 2: Graphical Representation of connections between belief revision, nonmonotonic inference, and conditional logics. A full line means there is a full correspondence between the two concepts, whereas a dashed line means that there is additional information needed for a full correspondence. For example, to define a belief revision operator on the basis of a nonmonotonic inference relation, one needs to additionally assume a context K which corresponds to ϕ, see e. g. [52].

problematise revision where instead of using Kleene's logic as a base logic, possibilistic logic, shown in [39] to be closely related to ADFs, as a base logic. Thereafter, in Section 9, we define and study dynamic conditionals based on such revisions. We compare our approach with related work in Section 10 and finally conlcude in Section 11.
Relation with previous work: A previous version of this paper was presented and published at the 18th International Conference on Principles of Knowledge Representation and Reasoning (KR'21) [35]. In addition to including the proofs of all results in that paper, the current version of this paper includes extensive new material. In more detail, the following is new material with respect to the previous version of this paper [35]:

- the study of revision of ADFs under two-valued semantics (Sections 4 and 5),
- the study of revision operators respecting strong equivalence (Section 7),
- the study of revision operators based on possibilistic logic (Section 8),
- the impossibility of reducing revision of ADFs to revision of acceptance conditions (Section 6.5),
- an extended discussion of related work (Section 10).

2 Preliminaries

In the following, we briefly recall some general preliminaries on propositional logic, as well as technical details on conditional logic and ADFs [8].

2.1 Propositional Logic

For a set At of atoms let $\mathcal{L}(\mathsf{At})$ be the corresponding propositional language constructed using the usual connectives $\wedge$ (*and*), $\vee$ (*or*), $\neg$ (*negation*) and $\rightarrow$ (*material implication*). A (classical) *interpretation* (also called *possible world*) ω for a propositional language $\mathcal{L}(\mathsf{At})$ is a function $\omega : \mathsf{At} \rightarrow \{\mathsf{T}, \mathsf{F}\}$. Let $\Omega(\mathsf{At})$ denote the set of all interpretations for At. We simply write Ω if the set of atoms is implicitly given. An interpretation ω *satisfies* (or is a *model* of) an atom $a \in \mathsf{At}$, denoted by $\omega \models a$, if and only if $\omega(a) = \mathsf{T}$. The satisfaction relation $\models$ is extended to formulas as usual. As an abbreviation we sometimes identify an interpretation ω with its *complete conjunction*, i. e., if $a_1, \ldots, a_n \in \mathsf{At}$ are those atoms that are assigned T by ω and $a_{n+1}, \ldots, a_m \in \mathsf{At}$ are those propositions that are assigned F by ω we identify ω by $a_1 \ldots a_n \overline{a_{n+1}} \ldots \overline{a_m}$ (or any permutation of this). For example, the interpretation ω_1 on $\{a, b, c\}$ with $\omega(a) = \omega(c) = \mathsf{T}$ and $\omega(b) = \mathsf{F}$ is abbreviated by $a\bar{b}c$. For $\Phi \subseteq \mathcal{L}(\mathsf{At})$ we also define $\omega \models \Phi$ if and only if $\omega \models \phi$ for every $\phi \in \Phi$. We define the set of models $\mathsf{Mod}(X) = \{\omega \in \Omega(\mathsf{At}) \mid \omega \models X\}$ for every formula or set of formulas X. A formula or set of formulas X_1 *entails* another formula or set of formulas X_2, denoted by $X_1 \vdash X_2$, if $\mathsf{Mod}(X_1) \subseteq \mathsf{Mod}(X_2)$.

	$\neg$	$\sim$	$\odot$
$\top$	$\bot$	$\bot$	$\bot$
u	u	$\top$	$\top$
$\bot$	$\top$	$\top$	$\bot$

$\wedge$	$\top$	u	$\bot$
$\top$	$\top$	u	$\bot$
u	u	u	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\vee$	$\top$	u	$\bot$
$\top$	$\top$	$\top$	$\top$
u	$\top$	u	u
$\bot$	$\top$	u	$\bot$

Table 1: Truth tables for connectives in Kleene's K

2.2 Kleenes Three-Valued Logic

Due to the three-valued nature of ADFs (see Section 2.5), we will need a three-valued logic to use as a basic logic underlying revision. Due to its high expressivity, we use Kleenes three-valued logic. A 3-valued interpretation for a set of atoms At is a function $v : \mathsf{At} \to \{\top, \bot, u\}$, which assigns to each atom in At either the value $\top$ (true, accepted), $\bot$ (false, rejected), or u (unknown). The set of all three-valued interpretations for a set of atoms At is denoted by $\mathcal{V}(\mathsf{At})$. We sometimes denote an interpretation $v \in \mathcal{V}(\{x_1, \ldots, x_n\})$ by $\dagger_1 \ldots \dagger_n$ with $v(x_i) = \dagger_i$ and $\dagger_i \in \{\top, \bot, u\}$, e.g., $\top\top$ denotes $v(a) = v(b) = \top$ for $\mathsf{At} = \{a, b\}$. A 3-valued interpretation v can be extended to arbitrary propositional formulas $\phi \in \mathcal{L}(\mathsf{At})$ via the truth tables in Table 1. We furthermore extend the language with a second, *weak negation* $\sim$, which is evaluated to true if the negated formula is false or undecided (i.e. there is no positive information for the negated formula). Thus, $\sim\phi$ means that no explicit information for ϕ being true ($v(\phi) \neq \top$) is given, whereas $\neg\phi$ means that ϕ is false ($v(\phi) = \bot$).

The truth table for $\sim$ can also be found in Table 1.[1]

It will prove convenient to define the connective $\odot$ which stipulates a formula is undecided. We define $\odot\phi = \sim(\neg\phi \vee \phi)$. We define $\mathcal{L}^{\mathsf{K}}(\mathsf{At})$ as the language based on At, the unary connectives $\langle \neg, \sim, \odot \rangle$ and the binary connectives $\langle \wedge, \vee, \to \rangle$.

The following facts about $\sim$, which show some similarities between $\sim$ and classical negation, will prove useful below:

Fact 1. For any $\phi \in \mathcal{L}^{\mathsf{K}}(\mathsf{At})$ and any $v \in \mathcal{V}(\mathsf{At})$: (1) $v(\sim\phi) \neq u$, and (2) $v(\sim\sim\phi) = \top$ iff $v(\phi) = \top$.

We can show that $\odot$ expresses the undecidedness of any formula $\phi \in \mathcal{L}^{\mathsf{K}}$:

Fact 2. For any $\phi \in \mathcal{L}^{\mathsf{K}}(\mathsf{At})$, $v(\odot\phi) = \top$ iff $v(\phi) = u$.

We define the set of three-valued interpretations that satisfy a formula $\phi \in \mathcal{L}^{\mathsf{K}}(\mathsf{At})$ as $\mathcal{V}(\phi) = \{v \in \mathcal{V}(\mathsf{At}) \mid v(\phi) = \top\}$. A formula X_1 K-*entails* another formula X_2, denoted $X_1 \models_{\mathsf{K}} X_2$, if $\mathcal{V}(X_1) \subseteq \mathcal{V}(X_2)$. $X_1 \equiv_{\mathsf{K}} X_2$ iff $X_1 \models_{\mathsf{K}} X_2$ and $X_2 \models_{\mathsf{K}} X_1$.

Given an interpretation $v \in \mathcal{V}(\mathsf{At})$, we define:

$$\mathsf{form}(v) = \bigwedge_{v(a)=\top} a \wedge \bigwedge_{v(a)=\bot} \neg a \wedge \bigwedge_{v(a)=u} \odot a$$

Clearly, $\mathsf{form}(v)$ expresses exactly the beliefs expressed by a three-valued interpretation:

Fact 3. For any $v \in \mathcal{V}(\mathsf{At})$ and any $a \in \mathsf{At}$: (1) $\mathsf{form}(v) \models_{\mathsf{K}} a$ iff $v(a) = \top$; (2) $\mathsf{form}(v) \models_{\mathsf{K}} \neg a$ iff $v(a) = \bot$; (3) $\mathsf{form}(v) \models_{\mathsf{K}} \odot a$ iff $v(a) = u$.

2.3 Defeasible Inference and Nonmonotonic Conditionals

When considering conditionals, we consider syntactic objects of the form $(\psi|\phi)$ (with $\phi, \psi \in \mathcal{L}$, which are read as "if ϕ is the case then typically ψ is the case as well". $(\mathcal{L}|\mathcal{L})$ is the set of all conditionals that can be formulated on the basis of $\mathcal{L}$. We follow the approach of de Finetti [15] who considered conditionals as *generalised indicator functions* for possible worlds resp. propositional interpretations ω:

$$((\psi|\phi))(\omega) = \begin{cases} 1 & : \quad \omega \models \phi \wedge \psi \\ 0 & : \quad \omega \models \phi \wedge \neg\psi \\ u & : \quad \omega \models \neg\phi \end{cases} \tag{1}$$

[1] In the terminology of [64], the negation $\sim$ corresponds to Bochvar's *external negation* [4] and $\neg$ corresponds to Kleene's negation in his three-valued logic. $\sim$ is also referred to as Kleene's *weak negation* [65], since the conditions for $\sim\phi$ being satisfied are weaker than those for $\neg\phi$ being satisfied (i.e. $\{\neg\phi\} \models_{\mathsf{K}} \sim\phi$).

where u stands for *unknown* or *indeterminate*. In other words, a possible world ω *verifies* a conditional $(\psi|\phi)$ iff it satisfies both antecedent ϕ and conclusion ψ ($(\psi|\phi)(\omega) = 1$); it *falsifies, or violates* $(\psi|\phi)$ iff it satisfies the antecedent but not the conclusion ($(\psi|\phi)(\omega) = 0$); otherwise the conditional is *not applicable*, i. e., the interpretation does not satisfy the antecedent ($(\psi|\phi)(\omega) = u$). We say that ω *satisfies* a conditional $(\psi|\phi)$ iff it does not falsify it, i. e., iff ω satisfies its *material counterpart* $\phi \to \psi$.

There are many different conditional logics (cf., e. g., [43, 53]), but a common idea underlying many semantics for nonmonotonic conditionals is that to validate the acceptance of a conditional $(\psi|\phi)$, it suffices to look whether its material counterpart $\phi \to \psi$ is validated in a subset of possible worlds. In this work, we will assume that a total preorder $\preceq \subseteq \Omega(\mathsf{At}) \times \Omega(\mathsf{At})$ over the set of possible worlds can be used to encode relevance of the possible worlds w.r.t. evaluation of conditionals. Such a preorder intuitively represents the relative plausibility of a world. In more detail, we will state that a conditional $(\psi|\phi)$ *is accepted in a context encoded by* $\preceq$ iff the consequent is validated by all $\preceq$-minimal worlds validating the antecedent ϕ, in symbols:

$$\mathsf{Mod}(\psi) \supseteq \min_{\preceq}(\mathsf{Mod}(\phi))$$

This is in full compliance with defeasible inference relations $\phi \mathrel{|\!\sim} \psi$ [49] expressing that from ϕ, ψ may be plausibly/defeasibly derived. We say that $\phi \preceq \psi$ iff $\omega \preceq \omega'$ for some $\omega \in \min_{\preceq}(\mathsf{Mod}(\phi))$ and some $\omega' \in \min_{\preceq}(\mathsf{Mod}(\psi))$. This allows for expressing the validity of defeasible inferences via stating that $\phi \mathrel{|\!\sim}_{\preceq} \psi$ iff $(\phi \wedge \psi) \prec (\phi \wedge \neg\psi)$. Thus, nonmonotonic conditionals as defined above can be seen as a syntactic counterpart to defeasible inference, in the sense that $(\psi|\phi)$ is accepted in a context encoded by $\preceq$ iff $\phi \mathrel{|\!\sim}_{\preceq} \psi$

An implementation of total preorders are *ordinal conditional functions (OCFs)*, (also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ [60]. They express degrees of (im)plausibility of possible worlds and propositional formulas ϕ by setting $\kappa(\phi) := \min\{\kappa(\omega) \mid \omega \models \phi\}$. Intuitively the implausibility of a formula ϕ is the minimal degree of implausibility $\kappa(\omega)$ of a world ω verifying this formula. OCFs κ provide a particularly convenient formal environment for nonmonotonic and conditional reasoning, allowing for simply expressing the acceptance of conditionals and nonmonotonic inferences via stating that $(\psi|\phi)$ is accepted by κ iff $\phi \mathrel{|\!\sim}_{\kappa} \psi$ iff $\kappa(\phi \wedge \psi) < \kappa(\phi \wedge \neg\psi)$, implementing formally the intuition of conditional acceptance based on plausibility mentioned above. For an OCF κ, $Bel(\kappa)$ denotes the propositional beliefs that are implied by all most plausible worlds, i. e. $Bel(\kappa) = \{\phi \mid \forall \omega \in \kappa^{-1}(0) : \omega \models \phi\}$. Specific examples of ranking models are system Z yielding the inference relation $\mathrel{|\!\sim}^{\mathrm{Z}}$ [30] and c-representations [41].

Example 2. Consider $\preceq$ defined over $\Omega(\{a, b, c\})$ as follows:

$$abc, \quad ab\overline{c}, \quad a\overline{b}c, \quad \overline{a}bc \quad \prec \quad a\overline{b}\overline{c}, \quad \overline{a}b\overline{c}, \quad \overline{a}\overline{b}c, \quad \overline{a}\overline{b}\overline{c}$$

Thus, for example, $\neg a \mathrel{|\!\sim}_{\preceq} b$, $\neg b \mathrel{|\!\sim}_{\preceq} a$, $\neg a \vee \neg b \mathrel{|\!\sim}_{\preceq} c$, $\top \mathrel{|\!\sim}_{\preceq} a \vee b$ and $a \mathrel{|\!\not\sim}_{\preceq} c$.

We recall some properties of conditional consequence relations introduced in the seminal work by Kraus, Lehman and Magidor (KLM) [43]:

(REF)	$\forall \phi \in \mathcal{L}(\mathsf{At})$:	$\phi \mathrel{|\!\sim} \phi$			
(CUT)	$\phi \mathrel{|\!\sim} \psi$	and	$\phi \wedge \psi \mathrel{|\!\sim} \gamma$	imply	$\phi \mathrel{|\!\sim} \gamma$
(CM)	$\phi \mathrel{|\!\sim} \psi$	and	$\phi \mathrel{|\!\sim} \gamma$	imply	$\phi \wedge \psi \mathrel{|\!\sim} \gamma$
(RW)	$\phi \mathrel{|\!\sim} \psi$	and	$\psi \models \gamma$	imply	$\phi \mathrel{|\!\sim} \gamma$
(LLE)	$\phi \equiv \psi$	and	$\psi \mathrel{|\!\sim} \gamma$	imply	$\phi \mathrel{|\!\sim} \gamma$
(OR)	$\phi \mathrel{|\!\sim} \gamma$	and	$\psi \mathrel{|\!\sim} \gamma$	imply	$(\phi \vee \psi) \mathrel{|\!\sim} \gamma$
(RM)	$\phi \mathrel{|\!\sim} \gamma$	and	$\phi \mathrel{|\!\not\sim} \neg\psi$	imply	$\phi \wedge \psi \mathrel{|\!\sim} \gamma$

We recall the following proposition, stating the connection between the KLM properties defined above and total preorders:

Proposition 1 ([50])**.** For any total preorder $\preceq$, $\mathrel{|\!\sim}_{\preceq}$ satisfies (REF), (CUT), (CM), (RW), (LLE), (OR) and (RM).

System Z is defined as follows. A conditional $(\psi|\phi)$ is tolerated by a finite set of conditionals Δ if there is a possible world ω with $(\psi|\phi)(\omega) = 1$ and $(\psi'|\phi')(\omega) \neq 0$ for all $(\psi'|\phi') \in \Delta$, i. e. ω verifies $(\psi|\phi)$ and does not falsify any (other) conditional in Δ. The Z-partitioning $(\Delta_0, \ldots, \Delta_n)$ of Δ is defined as:

- $\Delta_0 = \{\delta \in \Delta \mid \Delta \text{ tolerates } \delta\}$;

- $\Delta_1, \ldots, \Delta_n$ is the Z-partitioning of $\Delta \setminus \Delta_0$.

For $\delta \in \Delta$ we define: $Z_\Delta(\delta) = i$ iff $\delta \in \Delta_i$ and $(\Delta_0, \ldots, \Delta_n)$ is the Z-partioning of Δ. Finally, the ranking function κ^Z_Δ is defined via: $\kappa^Z_\Delta(\omega) = \max\{Z(\delta) \mid \delta(\omega) = 0, \delta \in \Delta\} + 1$, with $\max \emptyset = -1$. We can now define $\Delta \mathrel{|\!\sim}^{\mathrm{Z}} \phi$ iff $\phi \in Bel(\kappa^Z_\Delta)$.

2.4 Revising Propositional Formulas

We now recall the AGM-approach to belief revision [2] as reformulated for propositional formulas by [40]. The following postulates for revision operators $\star : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ are formulated:

(R1)	$\phi \star \psi \vdash \psi$
(R2)	If $\phi \wedge \psi$ is satisfiable, then $\phi \star \psi \equiv \psi \wedge \phi$
(R3)	If ψ is satisfiable, then so is $\phi \star \psi$
(R4)	If $\phi_1 \equiv \phi_2$ and $\psi_1 \equiv \psi_2$, $\phi_1 \star \psi_1 \equiv \phi_2 \star \psi_2$
(R5)	$(\phi \star \psi) \wedge \mu \vdash \phi \star (\psi \wedge \mu)$
(R6)	If $(\phi \star \psi) \wedge \mu$ is satisfiable, then $\phi \star (\psi \wedge \mu) \vdash (\phi \star \psi) \wedge \mu$

An important result is the semantical characterisation of such a belief revision operator. For such a characterisation, a function $f : \mathcal{L}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ that assigns to each propositional formula $\phi \in \mathcal{L}$ a total preorder $\preceq_\phi$ over $\Omega(\mathsf{At})$ is used. The revision of a formula ϕ by a formula ψ is then defined as the formula which has as models exactly the $\preceq_\phi$-minimal models that satisfy ψ.

Definition 1 ([40])**.** Given a formula $\phi \in \mathcal{L}(\mathsf{At})$, a function $f : \mathcal{L}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ assigning preorders $\preceq_\phi$ over $\Omega(\mathsf{At})$ to every formula $\phi \in \mathcal{L}(\mathsf{At})$ is faithful iff:

1. For every $\phi \in \mathcal{L}(\mathsf{At})$, if $\omega, \omega' \in \mathsf{Mod}(\phi)$ then $\omega \not\prec_\phi \omega'$,

2. For every $\phi \in \mathcal{L}(\mathsf{At})$, if $\omega \in \mathsf{Mod}(\phi)$ and $\omega' \notin \mathsf{Mod}(\phi)$ then $\omega \prec_\phi \omega'$,

3. For every $\phi, \phi' \in \mathcal{L}(\mathsf{At})$, if $\phi \equiv \phi'$ then $\preceq_\phi = \preceq_{\phi'}$.

In [40] the following representation theorem for an AGMrevision operator $\star$ was shown:

Theorem 1 ([40])**.** An operator $\star : \mathcal{L}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathcal{L}(\mathsf{At})$ satisfies R1–R6 iff there exists a faithful mapping $f^\star : \mathcal{L}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ that maps each formula $\phi \in \mathcal{L}(\mathsf{At})$ to a total preorder s.t.:

$$\mathsf{Mod}(\phi \star \psi) = \min_{f^\star(\phi)}(\mathsf{Mod}(\psi)) \tag{2}$$

2.5 Abstract Dialectical Frameworks

We briefly recall some technical details on ADFs following loosely the notation from [8]. An ADF D is a tuple $D = (\mathsf{At}, L, C)$ where At is a finite set of atoms, $L \subseteq \mathsf{At} \times \mathsf{At}$ is a set of links, and $C = \{C_s\}_{s \in \mathsf{At}}$ is a set of total functions (also called acceptance functions) $C_s : 2^{par_D(\mathsf{At})} \to \{\top, \bot\}$ for each $s \in \mathsf{At}$ with $par_D(s) = \{s' \in \mathsf{At} \mid (s', s) \in L\}$. An acceptance function C_s defines the cases when the statement s can be accepted (truth value $\top$), depending on the acceptance status of its parents in D. By abuse of notation, we will often identify an acceptance function C_s by its equivalent *acceptance condition* which models the acceptable cases as a propositional formula. In more detail, C_s expresses the conditions that are to be accepted for s to be accepted. $\mathfrak{D}(\mathsf{At})$ denotes the set of all ADFs $D = (\mathsf{At}, L, C)$.

Example 3. We consider the following ADF $D_1 = (\{a, b, c\}, L, C)$ (see also Example 1) with $L = \{(a, b), (b, a), (a, c), (b, c)\}$ and $C_a = \neg b$, $C_b = \neg a$ and $C_c = \neg a \vee \neg b$. Informally, the acceptance conditions can be read as "a is accepted if b is not accepted", "b is accepted if a is not accepted" and "c is accepted if a is not accepted or b is not accepted".

An ADF $D = (\mathsf{At}, L, C)$ is interpreted through 3-valued interpretations $\mathcal{V}(\mathsf{At})$ (see Section 2.2). Recall that $\Omega(\mathsf{At})$ consists of all the two-valued interpretations (i. e. interpretations such that for every $s \in \mathsf{At}$, $v(s) \in \{\top, \bot\}$). We define the information order $\leq_i$ over $\{\top, \bot, u\}$ by making u the minimal element: $u <_i \top$ and $u <_i \bot$ and this order is lifted pointwise as follows (given two valuations v, w over At): $v \leq_i w$ iff $v(s) \leq_i w(s)$ for every $s \in \mathsf{At}$. The set of two-valued interpretations extending a valuation v is defined as $[v]^2 = \{w \in \Omega \mid v \leq_i w\}$. Given a set of valuations V, we denote with $\sqcap_i V$ the valuation defined by $\sqcap_i V(s) = v(s)$ if for every $v' \in V$, $v(s) = v'(s)$ and $\sqcap_i V(s) = u$ otherwise. $\Gamma_D : \mathcal{V}(\mathsf{At}) \mapsto \mathcal{V}(\mathsf{At})$ is defined as $\Gamma_D(v)(s) = \sqcap_i [v]^2(C_s)$. Intuitively, $\Gamma_D(v)$ assigns to an atom s the consensus of the truth values assigned by all completions of v to C_s.

For the definition of the stable model semantics, we need to define the reduct D^v of D given v, defined as: $D^v = (\mathsf{At}^v, L^v, C^v)$ with: (1) $L^v = L \cap (\mathsf{At}^v \times \mathsf{At}^v)$, and (2) $C^v = \{C_s[\{\phi \mid v(\phi) = \bot\}/\bot] \mid s \in \mathsf{At}^v\}$, where $C_s[\phi/\psi]$ is the formula obtained by substituting every occurrence of ϕ in C_s by ψ.

Definition 2. Let $D = (\mathsf{At}, L, C)$ be an ADF with $v : \mathsf{At} \to \{\top, \bot, u\}$ an interpretation:

- v is a 2-valued model iff $v \in \Omega$ and $v(s) = v(C_s)$ for every $s \in \mathsf{At}$.
- v is *admissible for* D iff $v \leq_i \Gamma_D(v)$.
- v is *complete for* D iff $v = \Gamma_D(v)$.
- v is *preferred for* D iff v is $\leq_i$-maximally complete.
- v is *grounded for* D iff v is $\leq_i$-minimally complete.
- v is stable iff v is a model of D and $\{s \in \mathsf{At} \mid v(s) = \top\} = \{s \in \mathsf{At} \mid w(s) = \top\}$ where w is the grounded interpretation of D^v[2].

We denote by $\mathsf{2val}(D)$, $\mathsf{admissible}(D)$, $\mathsf{complete}(D)$, $\mathsf{prf}(D)$, $\mathsf{grounded}(D)$, respectively $\mathsf{stable}(D)$ the sets of two-valued, admissible, complete, preferred, grounded, respectively stable interpretations of D.

We finally define inference relations for ADFs:

Definition 3. Given $\mathsf{Sem} \in \{\mathsf{prf}, \mathsf{grounded}, \mathsf{2val}, \mathsf{stable}\}$, an ADF $D = (\mathsf{At}, L, C)$ and $\phi \in \mathcal{L}^{\mathsf{K}}(\mathsf{At})$ we define: $D \mathrel{|\!\!\sim}^{\cap}_{\mathsf{Sem}} \phi$ iff $v(\phi) = \top$ for all $v \in \mathsf{Sem}(D)$.

Example 4 (Example 3 continued)**.** The ADF of Example 3 has three complete models v_1, v_2, v_3 with:

$$\begin{array}{ccc} v_1(a) = \top & v_1(b) = \bot & v_1(c) = \top \\ v_2(a) = \bot & v_2(b) = \top & v_2(c) = \top \\ v_3(a) = u & v_3(b) = u & v_3(c) = u \end{array}$$

v_3 is the grounded interpretation whereas v_1 and v_2 are both preferred, two-valued and stable models.

It will be important to have characterisations of *realisability* of sets of interpretations under some semantics:

Definition 4. Given a set of atoms At, a set of interpretations $\mathcal{V} \subseteq \mathcal{V}(\mathsf{At})$ is realisable under semantics Sem iff there exists an ADF $D \in \mathfrak{D}(\mathsf{At})$ s.t. $\mathsf{Sem}(D) = \mathcal{V}$.

[55] shows that a set of interpretations is realisable under prf iff it is a $\leq_i$-antichain[3] whereas every (and only) singleton sets are realisable under $\mathsf{grounded}$:

Proposition 2 ([55])**.** Given a set of atoms At, (1) a set of interpretations $\mathcal{V} \subseteq \mathcal{V}(\mathsf{At})$ is realisable under prf iff $\mathcal{V} \neq \emptyset$ and for every $v, v' \in \mathcal{V}$, $v \not\leq_i v'$ and $v' \not\leq_i v$; (2) a set of interpretations $\mathcal{V} \subseteq \mathcal{V}(\mathsf{At})$ is realisable under $\mathsf{grounded}$ iff $\mathcal{V}$ has cardinality 1.

For the characterisation of realisability under the stable semantics we need to define the *truth-ordering* $\leq_\top$:

Definition 5. Given $v, v' \in \mathcal{V}(\mathsf{At})$, $v \leq_\top v'$ iff $\{s \in \mathsf{At} \mid v(s) = \top\} \subseteq \{s \in \mathsf{At} \mid v'(s) = \top\}$.

[2] [8] has show the grounded interpretation is uniquely defined for any ADF.

[3] Recall, a set of elements $\mathcal{V}'$ forms an antichain under an order $\leq$ if for no $v, v' \in \mathcal{V}'$, $v \leq v'$, i.e. all elements are pairwise incomparable under $\leq$.

It was shown in [61] that a set of interpretations is realisable under stable semantics iff it consists of two-valued interpretations and is a $\leq_\top$-antichain.

Proposition 3 ([61, Corollary 23])**.** A set of interpretations $\mathcal{V} \subseteq \mathcal{V}(\mathsf{At})$ is realisable under stable semantics iff $\mathcal{V} \subseteq \Omega(\mathsf{At})$ and for every $v, v' \in \mathcal{V}$, $v \not<_\top v'$.

A final result we will use is the fact that ADFs under two-valued model semantics are equi-expressive with propositional logic:

Theorem 2 ([61, Corollary 2])**.** Given a set of atoms At, $\{\mathsf{2mod}(D) \mid D \in \mathfrak{D}(\mathsf{At})\} = \wp(\Omega(\mathsf{At}))$

In other words, any subset of two-valued interpretations is realisable under the two-valued model semantics.

3 Revision of ADFs: Basic Idea

In this section, we explain and motivate the basic idea behind all types of revision treated in this paper. Informally, we are interested in revising argumentative contexts, which are represented by an ADF D, by new information, represented as logical formula ϕ, resulting in a revised argumentative context $D \star \phi$.

We concentrate on revising ADFs by formulas, resulting in a new ADF, i.e. revision operators $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$. Revisions will be always relative to a chosen semantics, and when this semantics is two-valued (e. g. two-valued models or stable models), we will restrict attention to revision by formulas in propositional logic in view of the two-valued nature of the mentioned semantics.

As an example of when this kind of revision can be useful, we refer to the travelling scenario from Example 1, represented by the ADF in Example 3.

In some works on revision of non-monotonic formalisms, including argumentation [47, 3, 59], revision of a knowledge base, e. g. a logic program or an argumentation framework, by another knowledge base is considered. For example, in [47], revision of an ADF by another ADF is studied. This can be easily done in our setting by representing the second ADF as the set of interpretations under Sem. As any set of two-valued interpretations $\mathcal{V} \subseteq \Omega(\mathsf{At})$ can be represented by the propositional formula $\bigvee \mathcal{V}$, and any set of three-valued interpretations $\mathcal{V} \subseteq \mathcal{V}(\mathsf{At})$ can be represented by the formula $\bigvee_{v \in \mathcal{V}} \mathsf{form}(v)$ in the language $\mathcal{L}^{\mathsf{K}}$ (see Fact 3), revision of ADFs by formulas in propositional logic respectively Kleene's logic (see Section 6) can handle revision of ADFs by other ADFs under two-valued respectively three-valued semantics.

4 Revision of ADFs under the Two-Valued Model Semantics

In this and the following sections, we introduce an approach to revision of ADFs under several semantics. In this section, we start with the arguably simplest case, namely the two-valued model semantics. As the general idea behind revision operators, and their semantics representation, is the same for all semantics, the case of two-valued models will serve as a gentle introduction to revision under the other, more complicated semantics.

As explained above, we first develop the theory of revision of ADFs under bivalent semantics. For this, it suffices to restrict attention to revision by formulas in classical propositional logic. We therefore adapt the AGM-postulates for propositional revision to the setting of revision-operators $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ of ADFs by propositional formulas as follows:

Definition 6. An operator $\star$ is a *bivalent ADF revision operator* (in short, $\mathsf{ADF}^2_\star$-operator) for an ADF $D = (\mathsf{At}, L, C)$ and a semantics Sem s.t. $\mathsf{Sem}(D) \subseteq \Omega(D)$[4] iff $\star$ satisfies:

- ($\mathsf{ADF}^2_\star 1$) $D \star \psi \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$
- ($\mathsf{ADF}^2_\star 2$) If $\mathsf{Sem}(D) \cap \mathsf{Mod}(\psi) \neq \emptyset$ then $\mathsf{Sem}(D \star \psi) = \mathsf{Sem}(D) \cap \mathsf{Mod}(\psi)$
- ($\mathsf{ADF}^2_\star 3$) If ψ is satisfiable, then $\mathsf{Sem}(D \star \psi) \neq \emptyset$
- ($\mathsf{ADF}^2_\star 4$) If $\mathsf{Sem}(D) = \mathsf{Sem}(D')$ and $\psi_1 \equiv \psi_2$, $\mathsf{Sem}(D \star \psi_1) = \mathsf{Sem}(D' \star \psi_2)$
- ($\mathsf{ADF}^2_\star 5$) $\mathsf{Sem}(D \star \psi) \cap \mathsf{Mod}(\mu) \subseteq \mathsf{Sem}(D \star (\psi \wedge \mu))$
- ($\mathsf{ADF}^2_\star 6$) If $\mathsf{Sem}(D \star \psi) \cap \mathsf{Mod}(\mu) \neq \emptyset$, then $\mathsf{Sem}(D \star \psi) \cap \mathsf{Mod}(\mu) \supseteq \mathsf{Sem}(D \star (\psi \wedge \mu))$

[4] The postulates ($\mathsf{ADF}^2_\star 1$)-($\mathsf{ADF}^2_\star 6$) can easily be generalised to a three-valued semantics by substituting $\mathsf{Sem}(D)$ by $\bigcup_{v \in \mathsf{Sem}(D)} [v]^2$. Since we define three-valued revisions below and for reasons of simplicity, we chose to restrict ourselves here to two-valued semantics.

Remark 1. An equivalent formulation of ($\mathsf{ADF}^2_\star 5$) respectively ($\mathsf{ADF}^2_\star 6$) that might be more intuitive to some readers is:

($\mathsf{ADF}^2_\star 5$) $\quad D \star \psi \mathrel{|\!\!\sim}^{\cap}_{\mathsf{2mod}} \mu \to \bigvee \mathsf{Sem}(D \star (\psi \wedge \mu))$[5]

($\mathsf{ADF}^2_\star 6$) $\quad$ If $\mathsf{Sem}(D \star \psi) \cap \mathsf{Mod}(\mu) \neq \emptyset$, then $D \star (\psi \wedge \mu) \mathrel{|\!\!\sim}^{\cap}_{\mathsf{2mod}} (\bigvee \mathsf{Sem}(D \star \psi) \wedge \mu)$

These postulates are explained as follows. $\mathsf{ADF}^2_\star 1$ requires that any revision is successful, i. e. the formula that induces the revision should follow from the revised ADF. The second postulate $\mathsf{ADF}^2_\star 2$ requires that if some of the Sem-interpretations of the original ADF satisfy the formula inducing the revision, the revised ADF should have as Sem-interpretations exactly the Sem-interpretations of the original ADF that satisfy the formula inducing the revision. The third postulate states that revising by a consistent formula results in a Sem-consistent ADF, i. e. an ADF that admits Sem-interpretations. $\mathsf{ADF}^2_\star 4$ requires syntax independence: revising ADFs with the same Sem-interpretations by equivalent formulas results in Sem-equivalent revised ADFs. Finally, $\mathsf{ADF}^2_\star 5$ and $\mathsf{ADF}^2_\star 6$ are direct adaptations of the *super-* and *sub-expansion postulates*. They require, in the non-trivial case where $D \star \psi \mathrel{|\!\!\not\sim}^{\cap}_{\mathsf{Sem}} \neg\mu$ (i. e. there is at least one Sem-interpretation of $D \star \psi$ that entails μ, or, in other words, $D \star \psi$ is consistent, under Sem, with μ), that the Sem-interpretations of $D \star (\psi \wedge \mu)$ are exactly the Sem-interpretations of $D \star \psi$ that satisfy μ.

We now semantically characterise revision of an ADF D with a formula ϕ in terms of total preorders over two-valued interpretations, in analogue to propositional revision. In more detail, we consider mappings of the type $\mathfrak{D}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$, i. e. functions mapping every ADF D to a total preorder $\preceq_D$ over possible worlds. We first modify Definition 1 of an assignment of preorders to be faithful w.r.t. an ADF D and a semantics Sem:

Definition 7. Given a semantics Sem s.t. $\mathsf{Sem}(D) \subseteq \Omega(\mathsf{At})$ for every $D \in \mathsf{At}$, a function $f : D \mapsto \preceq_D$ assigning[6] a total preorder $\preceq_D$ over $\Omega(\mathsf{At})$ to every ADF $D \in \mathfrak{D}(\mathsf{At})$ is faithful w.r.t. the semantics Sem iff:

1. For every $D \in \mathfrak{D}(\mathsf{At})$, if $\omega, \omega' \in \mathsf{Sem}(D)$, then $\omega \preceq_D \omega'$;
2. For every $D \in \mathfrak{D}(\mathsf{At})$, if $\omega \in \mathsf{Sem}(D)$ and $\omega' \notin \mathsf{Sem}(D)$, then $\omega \prec_D \omega'$;
3. For every $D, D' \in \mathfrak{D}(\mathsf{At})$, if $\mathsf{Sem}(D) = \mathsf{Sem}(D')$ then $\preceq_D = \preceq_{D'}$.

The intuition behind a faithful preorder for D (w.r.t. a two-valued semantics Sem) is that the beliefs justified on the basis of an ADF D can be represented as the formulas entailed by all interpretations in $\mathsf{Sem}(D)$ (which is in complete accordance with taking as beliefs all ϕ s.t. $D \mathrel{|\!\!\sim}^{\cap}_{\mathsf{Sem}} \phi$). A faithful preorder then represents the relative plausibility of formulas (or equivalently, possible worlds) given the ADF D. Therefore, the interpretations sanctioned by D are on the lowermost level, and other interpretations are ranked according to their plausibility by $\preceq_D$.

Example 5. We illustrate the above definitions by looking at the Dalal-revision operator [14], adapted here to our setting. We first define the symmetric distance function between two possible worlds $\omega, \omega' \in \Omega(\mathsf{At})$ as: $\omega \triangle \omega' = |s \in \mathsf{At} \mid \omega(s) \neq \omega'(s)|$. We can then define $\preceq^{\triangle}_D$ over $\Omega(\mathsf{At})$ by setting

$$\kappa_{\mathtt{dl}}(\omega) = \min\{\omega' \triangle \omega \mid \omega' \in \mathsf{2mod}(D)\}$$

for any $\omega \in \Omega(\mathsf{At})$ and letting $\omega_1 \preceq^{\triangle}_D \omega_2$ iff $\kappa_{\mathtt{dl}}(\omega_1) \leq \kappa_{\mathtt{dl}}(\omega_2)$.

For the ADF of Example 3, we then obtain the following ranking:

ω	$\kappa_{\mathtt{dl}}$	ω	$\kappa_{\mathtt{dl}}$	ω	$\kappa_{\mathtt{dl}}$	ω	$\kappa_{\mathtt{dl}}$
abc	1	$ab\overline{c}$	2	$a\overline{b}c$	0	$a\overline{b}\overline{c}$	1
$\overline{a}bc$	0	$\overline{a}b\overline{c}$	1	$\overline{a}\overline{b}c$	1	$\overline{a}\overline{b}\overline{c}$	2

We can now semantically characterise revision of an ADF D (under the two-valued semantics Sem) by a formula $\psi \in \mathcal{L}(\mathsf{At})$ as the ADF $D \star \psi$ s.t. :

$$\mathsf{Sem}(D \star \psi) = \min_{\preceq_D}(\mathsf{Mod}(\psi)) \tag{3}$$

[5] Recall that material implication $\to$ is defined as $\phi \to \psi := \neg\phi \vee \psi$.

[6] Recall that $\Omega(\mathsf{At})$ is the set of all (two-valued) interpretations for S.

Example 6. Looking again at Example 5, we can use Equation 3 to obtain a revision operator $\star_{\mathtt{dl}}$, which we illustrate by revising D with $\neg c$ based on the preorder $\kappa_{\mathtt{dl}}$ which has as two-valued models: $\mathsf{2mod}(D \star_{\mathtt{dl}} \neg c) = \{a\overline{b}\overline{c}, \overline{a}b\overline{c}\}$.

As we will see below, this revision satisfies all $\mathsf{ADF}^2_\star$-postulates.

Notice firstly that strictly speaking the revision above does not determine a unique ADF. However, it does determine a unique ADF *up to semantical equivalence* (see also Remark 2 below). Indeed, in view of Postulate $\mathsf{ADF}^2_\star 4$, we are justified in thus restricting our attention, since the result of the revision of two ADFs D_1 and D_2 with the same Sem-interpretations will result in two ADFs $D_1 \star \phi$ and $D_2 \star \phi$ with the same Sem-interpretations. Secondly, notice that the revision operator defined above is a purely semantical characterisation of revision of ADFs, i. e. the revision of an ADF D by a formula ψ is identified with a set of models. Below we will describe one strategy for obtaining a specific ADF on the basis of the set of two-valued models of an ADF.

We now proceed to the characterisation results of revision operators under two-valued model semantics. Since two-valued models are equi-expressive with propositional logic (Theorem 2), the proof strategy of the characterisation results is to simply establish an equivalence between revision of an ADF D by a formula ψ and revision of the propositional formula $\bigvee \mathsf{2mod}(D)$ by a formula ψ, and vice versa, establish equivalence between revision of a formula ϕ by a formula ψ and revision of an ADF D which has as two-valued models exactly $\mathsf{Mod}(\phi)$ by a formula ψ. We will see below that for revision under other semantics, this proof strategy does not work.

For the proofs, we first need to introduce some technicalities. Firstly, it will prove useful to have, given a set of possible worlds, a principled way of constructing an ADF that has as two-valued models exactly this set of possible worlds. This is done as follows (based on [61]):

Definition 8 ([61])**.** Given a set of models $\Lambda \subseteq \Omega$, $D_\Lambda = (\mathsf{At}, L, C)$ where for every $s \in \mathsf{At}$,

$$C_s = \Big(\bigvee_{\omega \in \Lambda, \omega(s)=\top} \bigwedge \omega\Big) \vee \Big(\bigvee_{\omega \in \Omega(\mathsf{At}) \setminus \Lambda, \omega(s)=\bot} \bigwedge \omega\Big)$$

Example 7. Let $\Lambda = \{\overline{a}bc, a\overline{b}c\}$. Then $D_\Lambda = (\{a, b, c\}, L, C)$ with:

$$\begin{array}{rcl} C_a & = & a\overline{b}c \vee (\overline{a}b\overline{c} \vee \overline{a}\overline{b}c \vee \overline{a}\overline{b}\overline{c}) \\ C_b & = & \overline{a}bc \vee (a\overline{b}\overline{c} \vee \overline{a}\overline{b}c \vee \overline{a}\overline{b}\overline{c}) \\ C_c & = & (\overline{a}bc \vee a\overline{b}c) \vee (ab\overline{c} \vee a\overline{b}\overline{c} \vee \overline{a}b\overline{c} \vee \overline{a}\overline{b}\overline{c}) \end{array}$$

We explain C_a in more detail as follows. The first disjunct $a\overline{b}c$ is obtained by taking the disjunction of all (complete conjunctions representing) interpretations in Λ that satisfy a, which is the single interpretation $a\overline{b}c$. Likewise, the second disjunction is obtained by taking the disjunction of all (complete conjunctions representing) $\overline{a}$-interpretations *not* in Λ. It can be checked that $\mathsf{2mod}(D_\Lambda) = \Lambda$ as desired (indeed, this follows from Proposition 4).

This construction is shown in [61] to be sound and complete:

Proposition 4 ([61, Theorem 1])**.** For any $\Lambda \subseteq \Omega(\mathsf{At})$, $\mathsf{2mod}(D_\Lambda) = \Lambda$.

Given a formula $\phi \in \mathcal{L}(\mathsf{At})$, we define $D[\phi]$ as the ADF s.t. $\bigvee \mathsf{2mod}(D[\phi]) \equiv \phi$ (notice that with Theorem 2 $D[\phi]$ exists for any $\phi \in \mathcal{L}(\mathsf{At})$). When this ADF is not unique we take $D_{\mathsf{Mod}(\phi)}$ as per Definition 8). As an example, $D[c \wedge ((\neg a \wedge b) \vee (a \wedge \neg b))] = D_\Lambda$ as in Example 7.

We now describe the basic strategy we follow: we start by specifying how to define a faithful function for propositional revision f_{form} in the sense of Definition 1 on the basis of a faithful function for ADF revision w.r.t. the two-valued model semantics (Definition 7), and vice versa we specify how to define a faithful function f_{ADF} for ADF revision w.r.t. the two-valued model semantics on the basis of a faithful function for ADF revision. Next, we show how to move from a revision operator $\star$ for ADFs to a revision operator $\langle\star\rangle$ for propositional logic, and vice versa, how to move from a revision operator $*$ for propositional logic to a revision operator $[*]$ for ADFs. The characterisation result for revision of ADFs then follows from the seminal characterisation result of [40] (here recalled as Theorem 1) for propositional revision. Graphically, we can summarise the proof strategy as follows:

$$\begin{tikzcd}
\text{Faithful mapping for ADF revision} \arrow[rr, leftrightarrow, dashed] \arrow[d, bend left, "f_{\mathsf{ADF}}"] & & \mathsf{ADF}^2_\star\text{-operator } \star \arrow[d, bend left, "\langle\cdot\rangle"] \\
\text{Faithful mapping for propositional revision} \arrow[u, bend left, "f_{\mathsf{form}}"] \arrow[rr, leftrightarrow, "\text{[40]}"] & & \text{Propositional revision operator } * \arrow[u, bend left, "[\cdot]"]
\end{tikzcd}$$

We now detail the constructions mentioned above. Given a function $f : \mathfrak{D}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ mapping ADFs D to total preorders $\preceq_D$, we define $f_{\mathsf{ADF}} : \mathcal{L}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ mapping propositional ϕ formulas to total preorders $\preceq_\phi$ as the function $f_{\mathsf{ADF}}(D) = f(\bigvee \mathsf{2mod}(D))$. Intuitively, f_{ADF} produces a faithful mapping for propositional revision by converting a total preorder faithful for D into a total preorder faithful for $\bigvee \mathsf{2mod}(D)$.

Likewise, given a function $f : \mathcal{L}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$, we define $f_{\mathsf{form}} : \mathcal{L}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ as $f_{\mathsf{form}}(\phi) = f(D[\phi])$. Intuitively, $f_{\mathsf{form}}(\phi)$ produces a faithful mapping for ADF revision by converting a total preorder faithful for ϕ into a total preorder faithful for $f(D[\phi])$. We illustrate here f_{form} (f_{ADF} works entirely analogously).

Example 8. Consider again $\preceq^{\Delta}_{D}$ from Example 5. Then $(\preceq^{\Delta}_{D})_{\mathsf{form}}$ produces a total preorder (identical to $(\preceq^{\Delta}_{D})_{\mathsf{form}}$) faithful for any ϕ with $\mathsf{Mod}(\phi) = \{\overline{a}bc, a\overline{b}c\}$. More in general, doing this for any $D \in \mathfrak{D}(\{a, b, c\})$ produces a faithful mapping (for propositional revision).

Not suprisingly, these two transformations preserve faithfulness. We first show the following useful lemma:

Lemma 1. Let an ADF D be given. Then $\omega \in \mathsf{Mod}(\bigvee \mathsf{2mod}(D))$ iff $\omega \in \mathsf{2mod}(D)$.

Proof. This can be seen immediately by observing that for any ADF D, $\mathsf{2mod}(D)$ is a set of complete conjunctions. □

Proposition 5. Let a set of atoms At be given. Then:

- Given some $f : \mathfrak{D}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$, if f is faithful w.r.t. $\mathsf{2mod}$ then f_{form} is faithful.
- Given some $f : \mathcal{L}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$, if f is faithful[7] then f_{ADF} is faithful w.r.t. $\mathsf{2mod}$.

Proof. We show the first item, by showing that if f is faithful w.r.t. $\mathsf{2mod}$, f_{form} satisfies the three items from Definition 1.

Ad 1. suppose $\omega, \omega' \in \mathsf{Mod}(\phi)$. This means that $\omega, \omega' \in \mathsf{Mod}(\bigvee \mathsf{2mod} D[\phi])$ (since $\phi \equiv \bigvee \mathsf{2mod} D[\phi]$). With Lemma 1 this means that $\omega, \omega' \in \mathsf{2mod}(D[\phi])$ and thus, since f is faithful w.r.t. $\mathsf{2mod}$, and by item 1 of Definition 7, $\omega' \not\prec \omega$. Thus, $\omega' \not\prec_{\mathsf{form}} \omega$.

Ad 2. Similar

Ad 3. Suppose some $\phi, \phi' \in \mathcal{L}(\mathsf{At})$ s.t. $\phi \equiv \phi'$. Since $\bigvee \mathsf{2mod}(D[\phi]) \equiv \phi$ and $\bigvee \mathsf{2mod}(D[\phi']) \equiv \phi'$. With Lemma 1, $\phi \equiv \phi'$ implies $\mathsf{2mod}(D[\phi]) = \mathsf{2mod}(D[\phi'])$. Since f is faithful w.r.t. $\mathsf{2mod}$, this means $\preceq_{D[\phi]} = \preceq_{D[\phi']}$. Thus, $f_{\mathsf{form}}(\phi) = \preceq_{D[\phi]} = \preceq_{D[\phi']} = f_{\mathsf{form}}(\phi')$.

The second item is analogous. □

We now detail the second pair of constructions, allowing us to move between revision operators for propositional logic and ADFs. Given a revision operator $\star$ for ADFs, we define the revision operator for propositional formulas as $\langle\star\rangle : \mathcal{L}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathcal{L}(\mathsf{At})$ as: $\phi\langle\star\rangle\psi = \bigvee \mathsf{2mod}(D[\phi] \star \psi)$. Likewise, given a propositional revision operator $*$, we define the revision operator for ADFs as $[*] : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ as: $D[*]\psi = D[\bigvee \mathsf{2mod}(D) * \psi]$.

Intuitively, given a revision operator $\star$ for ADFs, we obtain a revision operator $\langle\star\rangle$ for propositional logic as follows. To obtain the result for $\phi\langle\star\rangle\psi$, we first construct $D[\phi]$ to obtain an ADF with two-valued models $\mathsf{Mod}(\phi)$. We then apply the operator $\star$ to revise $D[\phi]$ by ψ, ending up with an ADF $D[\phi] \star \psi$. We then move back to the realm of propositional logic by letting $\phi \star \psi = \bigvee \mathsf{2mod}(D[\phi] \star \psi)$, i.e. $\phi \star \psi$ has as models the two-valued models of $D[\phi] \star \psi$. Constructing an operator for ADFs on the basis of a propositional revision operator is done analogously using $[\cdot]$.

[7] Recall defintion 1.

Example 9. Consider the revision operator for ADFs defined in Example 6 and the ADF D from Example 3. We show how to revise $c \wedge ((\neg a \wedge b) \vee (a \wedge \neg b))$ by $\neg c$ using $\langle \star_{\mathtt{d1}} \rangle$. We first construct $D[c \wedge ((\neg a \wedge b) \vee (a \wedge \neg b))]$, which, according to Example 7, we see is equal to D in Example 5. We then take $D \star_{\mathtt{d1}} \neg c$, which has, with Example 6, as two-valued models $\{a\overline{b}\overline{c}, \overline{a}b\overline{c}\}$. Thus, we obtain:

$$c \wedge ((\neg a \wedge b) \vee (a \wedge \neg b)) \langle \star_{\mathtt{d1}} \rangle \neg c = a\overline{b}\overline{c} \vee \overline{a}b\overline{c}.$$

Not surprisingly, these constructions preserve the respective postulates. We first note the following useful fact:

Fact 4. Given some ADFs $D, D' \in \mathfrak{D}(\mathsf{At})$, if $\bigvee \mathsf{2mod}(D) \equiv \bigvee \mathsf{2mod}(D')$ then $\mathsf{2mod}(D) = \mathsf{2mod}(D')$.

Proposition 6. Let a set of atoms At be given. Then:

- Let a propositional revision operator $* : \mathcal{L}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathcal{L}(\mathsf{At})$ be given. If $*$ satisfies (R1)-(R6) then $[*]$ satisfies ($\mathsf{ADF}^2_\star 1$)-($\mathsf{ADF}^2_\star 6$).
- Let a revision operator for ADFs $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ be given. If $\star$ satisfies ($\mathsf{ADF}^2_\star 1$)-($\mathsf{ADF}^2_\star 6$) then $\langle \star \rangle$ satisfies (R1)-(R6).

Proof. We first show the first item. We show that (Ri) implies ($\mathsf{ADF}^2_\star$i) for $1 \leq i \leq 4$. The cases for $i = 5, 6$ are analogous.

- ($\mathsf{ADF}^2_\star 1$): Notice that $D[*]\psi = D[\bigvee \mathsf{2mod}(D) * \psi]$, i.e. $\omega \in \mathsf{2mod}(D[*]\psi)$ iff $\omega \in \mathsf{Mod}(\bigvee \mathsf{2mod}(D) \star \psi)$. By (R1), $\bigvee \mathsf{2mod}(D) * \psi \vdash \psi$ and thus for every $\omega \in \mathsf{Mod}(\bigvee \mathsf{2mod}(D) * \psi)$, $\omega \models \psi$, i.e. for every $\omega \in \mathsf{2mod}(D[*]\psi)$, $\omega \models \psi$, which means $D[*]\psi \mid\!\sim^{\cap}_{\mathsf{2mod}} \psi$.
- ($\mathsf{ADF}^2_\star 2$): Suppose $\mathsf{2mod}(D) \cap \mathsf{Mod}(\psi) \neq \emptyset$. Thus, $\mathsf{Mod}(\bigvee \mathsf{2mod}(D)) \cap \mathsf{Mod}(\psi) \neq \emptyset$ and thus with (R2), $\bigvee \mathsf{2mod}(D) * \psi = \bigvee \mathsf{2mod}(D) \wedge \psi$. Thus $D[*]\psi = D[\bigvee \mathsf{2mod}(D) * \psi] = D[\bigvee \mathsf{2mod}(D) \wedge \psi]$, i.e. $\mathsf{2mod}(D[*]\psi) = \mathsf{Mod}(\bigvee \mathsf{2mod}(D) \wedge \psi)$. Thus, $\mathsf{2mod}(D[*]\psi) = \mathsf{2mod}(D) \cap \mathsf{Mod}(\psi)$.
- ($\mathsf{ADF}^2_\star 3$): Suppose ψ is satisfiable. By (R3), $\bigvee \mathsf{2mod}(D) * \psi$ is satisfiable. Since $D[*]\psi = D[\bigvee \mathsf{2mod}(D) * \psi]$, $\mathsf{2mod}(D[*]\psi) = D[\bigvee \mathsf{2mod}(D) * \psi] \neq \emptyset$.
- ($\mathsf{ADF}^2_\star 4$): Suppose $\mathsf{2mod}(D) = \mathsf{2mod}(D')$ and $\psi_1 \equiv \psi_2$. This means that $\bigvee \mathsf{2mod}(D) \equiv \bigvee \mathsf{2mod}(D')$. By (R4), $\bigvee \mathsf{2mod}(D) * \psi_1 \equiv \bigvee \mathsf{2mod}(D') * \psi_2$. Since $D[*]\psi_1 = D[\bigvee \mathsf{2mod}(D) * \psi_1]$ and $D'[*]\psi_2 = D'[\bigvee \mathsf{2mod}(D') * \psi_2]$, $\mathsf{2mod}(D * \psi_1) = \mathsf{2mod}(D' * \psi_2)$.

We now show the second item (again for the first four sub-items).

- (R1): Notice that $\phi\langle\star\rangle\psi = \bigvee \mathsf{2mod}(D[\phi] \star \psi)$. Since $\star$ satisfies ($\mathsf{ADF}^2_\star 2$), $D[\phi] \star \psi \mid\!\sim^{\cap}_{\mathsf{2mod}} \psi$, i.e. for every $\omega \in \mathsf{2mod}(D[\phi])$, $\omega \models \psi$. Thus, $\bigvee \mathsf{2mod}(D[\phi] \star \psi) \vdash \psi$ and thus $\phi\langle\star\rangle\psi \vdash \psi$.
- (R2): Suppose $\phi \wedge \psi$ is satisfiable. Since $\bigvee \mathsf{2mod}(D[\phi]) \equiv \phi$, $\mathsf{2mod}(D[\phi]) \cap \phi \neq \emptyset$ (indeed suppose towards a contradiction $\mathsf{2mod}(D[\phi]) \cap \mathsf{Mod}(\phi) = \emptyset$. Then there is no $\omega \in \mathsf{2mod}(D[\phi])$ s.t. $\omega \in \mathsf{Mod}(\phi)$. Since $\mathsf{2mod}(D[\phi]) = \mathsf{Mod}(\phi)$, this would contradict $\psi \wedge \phi$ being satisfiable.). Thus, by ($\mathsf{ADF}^2_\star 2$), $\mathsf{2mod}(D[\phi] \star \psi) = \mathsf{2mod}(D) \cap \mathsf{Mod}(\psi)$. Since $\phi\langle\star\rangle\psi = \bigvee \mathsf{2mod}(D[\phi] \star \psi)$, $\mathsf{Mod}(\phi\langle\star\rangle\psi) = \mathsf{2mod}(D[\phi]) \cap \mathsf{Mod}(\psi)$ and thus $\phi\langle\star\rangle\psi = \bigvee \mathsf{2mod}(D[\phi] \wedge \psi) \equiv \phi \wedge \psi$.
- (R3): Suppose ψ is satisfiable. By ($\mathsf{ADF}^2_\star 3$), $\mathsf{2mod}(D[\phi] \star \psi) \neq \emptyset$. Since $\phi\langle\star\rangle\psi = \bigvee \mathsf{2mod}(D[\phi] \star \psi)$, this means $\phi\langle\star\rangle\psi$ is satisfiable.
- (R4): Suppose $\phi_1 \equiv \phi_2$ and $\psi_1 \equiv \psi_2$. Thus, $\bigvee \mathsf{2mod}(D[\phi_1]) \equiv \bigvee \mathsf{2mod}(D[\phi_2])$. By Fact 4, $\mathsf{2mod}(D[\phi_1]) = \mathsf{2mod}(D[\phi_2])$. With ($\mathsf{ADF}^2_\star 4$) this implies that $\mathsf{2mod}(D \star \psi_1) = \mathsf{2mod}(D \star \psi_2)$ and thus $\phi_1[\star]\psi_1 = \bigvee \mathsf{2mod}(D[\phi_1] \star \psi_1) \equiv \phi_2[\star]\phi_2 = \bigvee \mathsf{2mod}(D[\phi_2] \star \psi_2)$. □

We can now show that revision operators can be characterised by means of a faithful mapping of ADFs to total preorders:

Corollary 1. Given a finite set of atoms At, an operator $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathcal{L}(\mathsf{At})$ is an $\mathsf{ADF}^2_\star$-operator for two-valued model semantics $\mathsf{2mod}$ iff there exists a function $f : \mathfrak{D}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ that is faithful w.r.t. $\mathsf{2mod}$ s.t.:

$$\mathsf{2mod}(D \star \psi) = \min_{\preceq_D}(\mathsf{Mod}(\psi)) \tag{4}$$

Proof. We first show the $\Rightarrow$-direction. Suppose there is a revision operator $\star$ for an ADF that satisfies ($\mathsf{ADF}^2_\star 1$)-($\mathsf{ADF}^2_\star 6$). By Proposition 6, $\langle\star\rangle$ satisfies (R1)-(R6). By Theorem 1, there is a function f that maps each formula $\phi \in \mathcal{L}(\mathsf{At})$ to a total preorder $\preceq_\phi$ that is faithful s.t. $\mathsf{Mod}(\phi \star \psi) = \min_{\preceq_\phi}(\mathsf{Mod}(\psi))$. By Proposition 5, f_{ADF} is faithful w.r.t. 2mod. Since $\langle\star\rangle$ satisfies (R1)-(R6), $[\langle\star\rangle]$ satisfies ($\mathsf{ADF}^2_\star 1$)-($\mathsf{ADF}^2_\star 6$) (with Proposition 6). Since for any ADF D and formula $\psi \in \mathcal{L}(\mathsf{At})$, $D[\langle\star\rangle]\psi = D[\bigvee \mathsf{2mod}(D)\langle\star\rangle\psi] = D[\bigvee \mathsf{2mod}(D[\bigvee \mathsf{2mod}(D) \star \psi])]$. Since $D[\bigvee \mathsf{2mod}(D) \star \psi] = D \star \psi$, $D[\langle\star\rangle]\psi = D[\bigvee \mathsf{2mod}(D \star \psi)]$ and thus $D[\langle\star\rangle]\psi = D \star \psi$.

We now show the $\Leftarrow$-direction. Suppose there is a function $f : \mathfrak{D}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ faithful w.r.t. 2mod. By Proposition 4, f_{form} is faithful. By Theorem 1, there is a revision operator that satifies R1 − R6 s.t. $\mathsf{Mod}(\phi \star \psi) = \min_{\preceq_\phi}(\mathsf{Mod}(\psi))$. By Proposition 6, $[\star]$ satisfies ($\mathsf{ADF}^2_\star 1$)-($\mathsf{ADF}^2_\star 6$). Since $\mathsf{2mod}(D[\star]\psi) = \mathsf{Mod}(\bigvee \mathsf{2mod}(D) \star \psi)$, we see that $\mathsf{2mod}(D[\star]\psi) = \min_{f_{\mathsf{form}}(D[\bigvee \mathsf{2mod}(D)])} \mathsf{Mod}(\psi)$. Since $f_{\mathsf{form}}(D[\bigvee \mathsf{2mod}(D)]) = f(D)$, $\min_{f_{\mathsf{form}}(D[\bigvee \mathsf{2mod}(D)])} \mathsf{Mod}(\psi) = \min_{f(D)}(\mathsf{Mod}(\psi))$ and thus we have shown that $\mathsf{2mod}(D[\star]\psi) = \min_{f(D)}(\mathsf{Mod}(\psi))$. ☐

In conclusion, the results above show that revision of ADFs under the two-valued semantics is equivalent to revision of propositional logical formulas. In the next sections, we will see that this is not the case for revision of ADFs under other semantics.

Remark 2. As mentioned above, revision is characterised here purely semantically, in the sense that a revised ADF is only specified by its two-valued models. In other words, we only know what $\mathsf{2mod}(D \star \phi)$ looks like, and not how the acceptance conditions of $D \star \phi$ look like. Our paper is not the only work with this feature, as in fact most works on revision of non-monotonic formalisms share this semantical nature (e. g. [11, 3, 20, 46, 59]). Furthermore, in view of the postulate $\mathsf{ADF}^2_\star 4$, it is sufficient to specify $\mathsf{Sem}(D \star \phi)$. However, it is not hard to specify a method to obtain specific acceptance conditions on the basis of a set of two-valued models.

For the two-valued model semantics, with Theorem 2, we can then syntactically characterise this set of models by an ADF. One can, for example, construct such an ADF as follows: we illustrate this procedure with Example 5:

Example 10. Revising the ADF D from Example 5 by $\neg c$ resulted in $\mathsf{2mod}(D \star \neg c) = \{a\overline{b}\overline{c}, \overline{a}b\overline{c}\}$ as we saw in Example 6. We now apply Definition 8 to obtain:

$$\begin{array}{rcl} C'_a & = & (a\overline{b}\overline{c} \vee \overline{a}bc) \vee (\overline{a}\overline{b}c \vee \overline{a}\overline{b}\overline{c}) \\ C'_b & = & (\overline{a}b\overline{c}) \vee (\overline{a}\overline{b}c \vee \overline{a}\overline{b}c) \\ C'_c & = & \bigvee \emptyset \vee (ab\overline{c} \vee \overline{a}\overline{b}\overline{c}) \end{array}$$

C'_c can be further simplified (in view of $\bigvee \emptyset \equiv \bot$):

$$C''_c \quad = \quad (ab\overline{c} \vee \overline{a}\overline{b}\overline{c})$$

We can then set the ADF $D' = (\{a, b, c\}, L, \{C'_a, C'_b, C''_c\})$ as one possible representative of $D \star \neg c$.

5 Revision of ADFs under the Stable Semantics

This section contains a characterisation of revisions under stable semantics by a class of total preorders. The basic idea is that every "layer" is a $\leq_\top$-antichain. This ensures that every $\preceq_D$-minimal set of two-valued interpretations is realisable under the stable semantics in view of Proposition 3. The need for this requirement is shown by the following example

Example 11. Consider the ADF D from Example 3 and consider $\preceq$ defined as:

$$\overline{a}bc, a\overline{b}c \prec abc, \overline{a}b\overline{c}, \overline{a}\overline{b}c, a\overline{b}\overline{c} \prec \ldots.$$

Notice that $\preceq$ is faithful w.r.t. stable. If we revise by $ab \vee \neg c$ by selecting the $\preceq$-minimal models satisfying $ab \vee \neg c$, we obtain $\mathsf{stable}(D \star (ab \vee \neg c)) = \{abc, \overline{a}b\overline{c}, a\overline{b}\overline{c}\}$. However, there exists no ADF $(D \star (ab \vee \neg c) \in \mathfrak{D}(\{a, b, c\})$ with $\{abc, \overline{a}b\overline{c}, a\overline{b}\overline{c}\}$ as stable models, since, $\overline{a}b\overline{c} <_\top abc$ contradicts $\mathsf{stable}(D \star (ab \vee \neg c))$ forming an $<_\top$-antichain (which we know in view of Proposition 3).

This problematic behaviour can be avoided by requiring additionally that every layer of a faithful mapping is an $\leq_\top$-antichain:

Definition 9. Given a semantics Sem s.t. $\mathsf{Sem}(D) \subseteq \Omega(\mathsf{At})$ for every $D \in \mathfrak{D}(\mathsf{At})$, a function $f : D \mapsto \preceq_D$ assigning a total preorder $\preceq_D$ over $\Omega(\mathsf{At})$ to every ADF $D \in \mathfrak{D}(\mathsf{At})$ is a $\top$-*modular faithful assignment* w.r.t. the semantics Sem iff:

1. if $\omega_1 \preceq_D \omega_2$ and $\omega_2 \preceq_D \omega_1$ then $\omega_1 \not<_\top \omega_2$ and $\omega_2 \not<_\top \omega_1$;
2. For every $D \in \mathfrak{D}(\mathsf{At})$, if $\omega, \omega' \in \mathsf{Sem}(D)$ then $\omega' \preceq_D \omega$;
3. for every $D \in \mathfrak{D}(\mathsf{At})$, if $\omega \in \mathsf{Sem}(D)$ and $\omega' \notin \mathsf{Sem}(D)$ then $\omega \prec_D \omega'$;
4. for every $D, D' \in \mathfrak{D}(\mathsf{At})$, if $\mathsf{Sem}(D) = \mathsf{Sem}(D')$ then $\preceq_D = \preceq_{\mathsf{Sem}(D')}$ for any ADF $D' = (\mathsf{At}, L', C')$.

Thus, the above definition extends faithful mappings with the requirement that every layer is $\leq_t$-modular.

Example 12. Consider again the preorder $\preceq$ from Example 11. We can turn this into a T-modular faithful mapping $\preceq'$ as follows (among many other possibilities):

$$\overline{a}bc, a\overline{b}c \prec' abc \prec' \overline{a}b\overline{c}, \overline{a}\overline{b}c, a\overline{b}\overline{c} \prec' \dots.$$

Revising D by $ab \vee \neg c$ now results in $\mathsf{stable}(D \star (ab \vee \neg c)) = \{abc\}$. By Proposition 3, $\{abc\}$ is realisable under stable semantics. This illustrates the usefulness of $\top$-modular faithful mappings, as now any selection is ensured to be realisable under stable semantics. This is further ilustrated by the following propositions.

Proposition 7. Let a set of atoms At be given. Then, if $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ is an $\mathsf{ADF}^2_\star$-operator for stable, there is a $\top$-modular faithful mapping $f : D \mapsto \preceq_D$ s.t. $\mathsf{stable}(D \star \psi) = \min_{\preceq_D}(\mathsf{Mod}(\psi))$.

Proof. Suppose At is a set of atoms and $\star : \mathfrak{D}(\mathsf{At}) \times \mathfrak{D}(\mathsf{At}) \to \mathcal{L}(\mathsf{At})$ is an $\mathsf{ADF}^2_\star$-operator for stable. For any $D \in \mathfrak{D}(\mathsf{At})$ we define $\preceq_D$ as follows: $\omega \preceq_D \omega'$ iff $\omega \in \mathsf{stable}(D \star \mathsf{form}(\omega) \vee \mathsf{form}(\omega'))$. Showing that $\preceq_D$ is a total preorder is done entirely analogous as in [40, Theorem 3.3]. We show now that $\preceq_D$ is $\top$-modular faithful mapping:

1. We now show the first condition of Definition 9. Suppose that there are some $\omega, \omega' \in \Omega(\mathsf{At})$ s.t. $\omega <_\top \omega'$. By Proposition 3, $\omega \notin \mathsf{stable}(D \star \mathsf{form}(\omega) \vee \mathsf{form}(\omega'))$ or $\omega' \notin \mathsf{stable}(D \star \mathsf{form}(\omega) \vee \mathsf{form}(\omega'))$ (and not both). With ($\mathsf{ADF}^2_\star 1$), $\omega \in \mathsf{stable}(D \star \mathsf{form}(\omega) \vee \mathsf{form}(\omega'))$ or $\omega' \in \mathsf{stable}(D \star \mathsf{form}(\omega) \vee \mathsf{form}(\omega'))$. Thus, $\omega \prec_D \omega'$ or $\omega' \prec_D \omega$.
2. We now show the second condition of Definition 9. Suppose $\omega \in \mathsf{stable}(D)$. By ($\mathsf{ADF}^2_\star 2$), $\omega \in \mathsf{stable}(D \star \mathsf{form}(\omega) \vee \mathsf{form}(\omega'))$ for any $\omega' \in \Omega(\mathsf{At})$ and thus $\omega' \not\prec_D \omega$, and thus, in view of the totality of $\preceq_D$, $\omega \preceq_D \omega'$ for any such ω'.
3. We now show the third condition of Definition 9. Suppose $\omega \in \mathsf{stable}(D)$ and $\omega' \notin \mathsf{stable}(D)$. By ($\mathsf{ADF}^2_\star 2$), $\{\omega\} = \mathsf{stable}(D \star \mathsf{form}(\omega) \vee \mathsf{form}(\omega'))$ and thus $\omega \prec_D \omega'$.
4. We finally show the fourth condition of Definition 9. Suppose $\mathsf{stable}(D) = \mathsf{stable}(D')$ for some $D, D' \in \mathfrak{D}(\mathsf{At})$. With ($\mathsf{ADF}^2_\star 4$), for any $\psi \in \mathcal{L}(\mathsf{At})$, $\mathsf{stable}(D \star \psi) = \mathsf{stable}(D' \star \psi)$ and thus by definition of $\preceq_D$ respectively $\preceq_{D'}$, $\preceq_D = \preceq_{D'}$. □

Proposition 8. Let a finite set of atoms At and a $\top$-modular faithful mapping $f : D \mapsto \preceq_D$[8] be given. If $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ is defined by $\mathsf{stable}(D \star \psi) = \min_{f(D)}(\mathsf{Mod}(\psi))$, then $\star$ is a $\mathsf{ADF}^2_\star$-operator for stable.[9]

Proof. We show that $\star$ is well-defined, showing that $\star$ satisfies ($\mathsf{ADF}^2_\star 1$)-($\mathsf{ADF}^2_\star 6$) is done entirely analogous as in [40, Theorem 3.3]. To show that $\star$ is well-defined (i.e. $\min_{f(D)}(\mathsf{Mod}(\psi)) \neq \emptyset$ and is realisable under stable), with Proposition 3 it suffices to show that for every $D \in \mathfrak{D}(\mathsf{At})$ and $\psi \in \mathcal{L}(\mathsf{At})$, $\min_{\preceq_D}(\mathsf{Mod}(\psi)) \neq \emptyset$ and $\min_{\preceq_D}(\mathsf{Mod}(\psi))$ is a $<_\top$-antichain. The former can be easily seen by the fact that $\preceq_D$ is a total preorder on a finite set $\Omega(\mathsf{At})$. For the latter, suppose that $\omega, \omega' \in \min_{\preceq_D}(\mathsf{Mod}(\psi))$ for some arbitrary $\psi \in \Omega(\mathsf{At})$. Since $\preceq_D$ is $\top$-modular faithful, $v_1 \not<_\top v_2$ and $v_2 \not<_\top v_1$. Thus, it is reasonable under stable. □

[8] Recall that a $\top$-modular faithful mapping assigns a *total* preorder to every ADF D.

[9] Strictly speaking, $\mathsf{stable}(D \star \psi) = \min_{f(D)}(\mathsf{Mod}(\psi))$ does not define an ADF, but is merely a constraint on the ADF $D \star \psi$. However, as also explained in Remark 2, we can simply assume a construction method for such an ADF, as the $\mathsf{ADF}^2_\star 4$-postulate does not allow to distinguish between two ADFs with the same stable interpretations anyway.

We can now show the following theorem as a corollary of Propositions 7 and 8:

Theorem 3. An operator $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathcal{L}(\mathsf{At})$ is a revision operator $\star$ for stable semantics iff there exists a function $f : \mathfrak{D}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ that is $\top$-modular faithful w.r.t. stable s.t.:

$$\mathsf{stable}(D \star \psi) = \min_{\preceq_D}(\mathsf{Mod}(\psi)) \tag{5}$$

6 ADF-revisions for trivalent semantics

In this section, we study revision of ADFs by formulas under trivalent semantics, i. e. semantics for ADFs where the interpretations selected by the semantics can be three-valued, such as the admissible, complete, preferred and grounded semantics. We define in Section 6.1 postulates for revision operators under trivalent semantics. In Section 6.1, we show that these postulates cannot be satisfied under the admissible and complete semantics. Positive results are given in Sections 6.3 and 6.4, where we characterise revision operators in terms of total preorders over three-valued interpretations for the preferred semantics (Section 6.3) and the grounded semantics (Section 6.4).

6.1 ADF-revision under trivalent semantics: postulates and semantics

In this section we define a new approach to revision of ADFs for three-valued semantics. In more detail, we define an operator $\star$ that allows to revise an ADF (under some three-valued semantics) by a formula in the language $\mathcal{L}^{\mathsf{K}}$.[10] In other words, the type of a revision operator $\star$ under trivalent semantics is $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$. The idea is basically the same as revision of ADFs under two-valued semantics, only that we now allow for revision by formulas in an extended language $\mathcal{L}^{\mathsf{K}}$. The reason we do this is to "match" the additional expresiveness allowed for by three-valued semantics. Indeed, we can now revise e. g. by formulas expressing that a formula is undecided or false ($\sim \phi$) or that a formula is undecided ($\odot\phi$).

We adapt the AGM-postulates for propositional revision described in Section 2.4 to revision operators for ADFs in the following way:

Definition 10. An operator $\star$ is a *trivalent ADF revision operator* (in short, $\mathsf{ADF}^3_\star$-operator) for a semantics Sem iff $\star$ satisfies (for any $\phi, \psi, \mu \in \mathcal{L}^{\mathsf{K}}$):
($\mathsf{ADF}^3_\star 1$) $D \star \psi \mathrel{|\!\sim}^{\cap}_{\mathsf{Sem}} \psi$.[11]
($\mathsf{ADF}^3_\star 2$) If $\mathsf{Sem}(D) \cap \mathcal{V}(\psi) \neq \emptyset$ then $\mathsf{Sem}(D \star \psi) = \mathsf{Sem}(D) \cap \mathcal{V}(\psi)$.
($\mathsf{ADF}^3_\star 3$) If $\mathcal{V}(\psi) \neq \emptyset$ then $\mathsf{Sem}(D \star \psi) \neq \emptyset$.
($\mathsf{ADF}^3_\star 4$) If $\mathsf{Sem}(D) = \mathsf{Sem}(D')$ and $\psi \equiv_{\mathsf{K}} \psi'$ then $\mathsf{Sem}(D \star \psi) = \mathsf{Sem}(D' \star \psi')$.
($\mathsf{ADF}^3_\star 5$) $\mathsf{Sem}(D \star \psi) \cap \mathcal{V}(\mu) \subseteq \mathsf{Sem}(D \star (\psi \wedge \mu))$.
($\mathsf{ADF}^3_\star 6$) If $\mathsf{Sem}(D \star \psi) \cap \mathcal{V}(\mu) \neq \emptyset$, then $\mathsf{Sem}(D \star (\psi \wedge \mu)) \subseteq \mathsf{Sem}(D \star \psi) \cap \mathcal{V}(\mu)$.

The motivation of these postulates is entirely the same as that of the postulates for bivalent revision operators (see Definition 6). Indeed, these postulates are identical to those defined in Definition 6 besides the fact that we now consider formulas in $\mathcal{L}^{\mathsf{K}}$ instead of propositional formulas.

The main question we answer in the rest of this section is whether $\mathsf{ADF}^3_\star$-operators can be characterised semantically analogously to propositional revision operators (Theorem 1). The central concept for such a characterisation will be that of a *faithful mapping* of ADFs to total preorders over $\mathcal{V}(\mathsf{At})$, i. e. a mapping of the type $\mathfrak{D}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$:

Definition 11. Given a semantics Sem and an ADF $D = (\mathsf{At}, L, C)$, a mapping $f : D \mapsto \preceq_D$ associating a total preorder $\preceq_D$ to every ADF D is a *faithful mapping* for semantics Sem if, for every $D \in \mathfrak{D}(\mathsf{At})$ and for every $v_1, v_2 \in \mathcal{V}(\mathsf{At})$:

1. if $v_1, v_2 \in \mathsf{Sem}(D)$ then $v_1 \preceq_D v_2$; and
2. if $v_1 \in \mathsf{Sem}(D)$ and $v_2 \notin \mathsf{Sem}(D)$ then $v_1 \prec_D v_2$; and

[10] Recall, $\mathcal{L}^{\mathsf{K}}(\mathsf{At})$ is the language based on At, the unary connectives $\langle \neg, \sim, \odot \rangle$ and the binary connectives $\langle \wedge, \vee, \to \rangle$
[11] Or, equivalently, $\mathsf{Sem}(D \star \psi) \subseteq \mathcal{V}(\psi)$.

3. if $\mathsf{Sem}(D) = \mathsf{Sem}(D')$ then $\preceq_D = \preceq_{D'}$.

This definition is again entirely analogous to Definition 7, except for the fact that the resulting total preorders range over the three-valued interpretations $\mathcal{V}(\mathsf{At})$ instead of the two-valued interpretations $\Omega(\mathsf{At})$.

A faithful mapping is in general not sufficient to ensure a characterisation of $\mathsf{ADF}^3_\star$-operators. The main problem is that a faithful mapping does not ensure that a selection of $\preceq_D$-minimal interpretations that satisfy ϕ are realisable by some ADF $D \star \phi \in \mathfrak{D}(\mathsf{At})$ under the semantics under consideration. In the following subsections, we investigate whether and how such realisability can be ensured by imposing additional conditions on faithful mappings. We shall see (in Section 6.2) that in general, such conditions cannot be found, by showing that for admissible and complete semantics no $\mathsf{ADF}^3_\star$-operator satisfying all postulates exists. Thereafter, we shall provide conditions and corresponding characterisation theorems for preferred (Section 6.3) and grounded (Section 6.4) semantics.

6.2 Impossibility of Rational Revision under Admissible and Complete Semantics

In this section, we show that a revision operator that satisfies $\mathsf{ADF}^3_\star 1$-$\mathsf{ADF}^3_\star 6$ for the admissible or complete semantics does not exist. In particular, we show that no revision operator can satisfy $\mathsf{ADF}^3_\star 2$. A similar result can be found in [20, Proposition 2] for revision of abstract argumentation frameworks under complete semantics. Intuitively, the reason that no revision operator satisfying $\mathsf{ADF}^3_\star 2$ for these semantics exists is that not every subset of $\mathsf{Sem}(D)$ is realisable under Sem for $\mathsf{Sem} \in \{\mathsf{complete}, \mathsf{admissible}\}$. For example, a set not containg the interpretation that sets $v(s) = u$ for every $s \in \mathsf{At}$ is not realisable under admissible semantics. Thus, if we revise D by ϕ that is satisfied by exactly such a subset, $\mathsf{ADF}^3_\star 2$ forces $\mathsf{Sem}(D \star \phi)$ to equal a non-realisable set of interpretations.

Proposition 9. There is no operator $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ that satisfies $\mathsf{ADF}^3_\star 2$ for $\mathsf{Sem} = \mathsf{complete}$ or $\mathsf{Sem} = \mathsf{admissible}$.

Proof. We show that for the ADF $D = (\{a\}, L, \{C_a = a\})$ there exists no operator $\star$ s.t. $\mathsf{ADF}^3_\star 2$ is satisfied. Indeed, suppose towards a contradiction that $\mathsf{ADF}^3_\star 2$ holds for an operator $\star$ for $\mathsf{Sem} \in \{\mathsf{complete}, \mathsf{admissible}\}$. Notice that $\mathsf{complete}(D) = \{u, \top, \bot\} = \mathsf{admissible}(D)$. Consider the revision $D \star (a \vee \neg a)$. Since $\mathcal{V}(a \vee \neg a) = \{\top, \bot\}$, $\mathcal{V}(a \vee \neg a) \cap \mathsf{Sem}(D) \neq \emptyset$ for $\mathsf{Sem} \in \{\mathsf{complete}, \mathsf{admissible}\}$, and thus, with our supposition that $\mathsf{ADF}^3_\star 2$ holds for $\star$ under Sem, $\mathsf{Sem}(D \star (a \vee \neg a)) = \mathsf{Sem}(D) \cap \mathcal{V}(a \vee \neg a) = \{\top, \bot\}$. But there is no ADF $D \star (a \vee \neg a) \in \mathfrak{D}(\{a\})$ s.t. $\mathsf{Sem}(D \star (a \vee \neg a)) = \{\top, \bot\}$, i.e. the result of this revision is not realisable under Sem. To see this for $\mathsf{Sem} = \mathsf{admissible}$, it suffices to observe that $u \in \mathsf{admissible}(D')$ for any $D' \in \mathfrak{D}(\{a\})$. To see this for $\mathsf{Sem} = \mathsf{complete}$, it suffices to observe that there exists for any ADF a unique $\leq_i$-minimal complete extension [8]. However, $\{\top, \bot\}$ does not contain a unique $\leq_i$-minimal element. □

The above problem can be potentially circumvented by studying weaker versions of the $\mathsf{ADF}^3_\star 2$-postulate. We leave such investigations for future work.

6.3 Revision of ADFs under Preferred Semantics

In this section, we give a semantical characterisation of revision operators for preferred semantics, in terms of faithful mappings of ADFs to total preorders over three-valued interpretations. Like the case for stable semantics (recall Section 5), we need to ensure realisability of selections of $\preceq_D$-minimal interpretations under the preferred semantics. This is done by requiring that every layer of a total preorder is incomparable under the information ordering $\leq_i$ (recall Section 2.5). We call such mappings *i-modular faithful mappings* (imf-mappings). We first motivate the need for such mappings in an example and then define imf-mappings (Definition 12).

The following example shows that faithful mappings without any additional conditions do not always lead to a sound semantical characterisation of $\mathsf{ADF}^3_\star$-revision operators for preferred semantics:

Example 13. We show that a naive adaptation of Dalal's revision operator [14] can lead to selections unrealisable under the preferred semantics. We use the symmetric distance function $\triangle$ defined between truth-values as follows: $\top \triangle \bot = 1$, $\top \triangle u = \bot \triangle u = 0.5$ and $x \triangle x = 0$ for any $x \in \{\top, \bot, u\}$ (cf. [61]). We then lift this to interpretations $v, v' \in \mathcal{V}(\mathsf{At})$ as follows: $v \triangle v' = \Sigma_{s \in \mathsf{At}} v(s) \triangle v'(s)$. Defining a faithful preorder $\preceq_D$ based solely on this distance function (e. g. by setting $v_1 \preceq_D^{\mathsf{prf},\Delta} v_2$ iff $\min_{v \in \mathsf{prf}(D)}(v \Delta v_1) \leq \min_{v \in \mathsf{prf}(D)}(v \Delta v_2)$) would not result in a selection realisable under prf, since there could be $\preceq_D^{\mathsf{prf},\Delta}$-equal interpretations that are not $\leq_i$-incompatible.

Take e. g. the ADF D_1 from Example 3. Observe that $\mathsf{prf}(D) = \{\top\bot\top, \bot\top\top\}$ as well as $\top uu, \top\bot\bot \in \min_{\preceq_D^{\mathsf{prf},\Delta}} \mathcal{V}(a \wedge \sim b \wedge \sim c)$. Revising D with $a \wedge \sim b \wedge \sim c$ would thus result in an ADF $D_1 \star a \wedge \sim b \wedge \sim c$ which has $\top uu$ and $\top\bot\bot$ among its preferred extensions, which is impossible in view of Proposition 2, since $\top uu <_i \top\bot\bot$, i. e. the interpretations are not $\leq_i$-incompatible.

To avoid selections of interpretations that are non-realisable under preferred semantics like in Example 13, an additional condition on faithful mappings has to be imposed. This condition we call *i-modularity*, and requires that every $\preceq_D$-layer is an $\leq_i$-antichain, i. e. all interpretations in a $\preceq_D$-layer are $\leq_i$-incompatible. We will denote, for a preorder $\preceq$, $v \preceq v'$ and $v' \preceq v$ as $v \approx v'$. Formally, i-modular faithful mappings are of the type $\mathfrak{D}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$, i. e. associating a total preorder $\preceq_D$ with any D and defined as follows:

Definition 12. Given a semantics Sem and an ADF $D = (\mathsf{At}, L, C)$, a mapping $f : D \mapsto \preceq_D$ associating a total preorder $\preceq_D$ to every ADF D is an *i-modular faithful mapping* (imf-mapping) for semantics Sem if it is faithful w.r.t. Sem and for every $D \in \mathfrak{D}(\mathsf{At})$ and every $v_1, v_2 \in \mathcal{V}(\mathsf{At})$: if $v_1 \approx_D v_2$ then $v_1 \not<_i v_2$ and $v_2 \not<_i v_1$.

In Theorem 4 we show that $\mathsf{ADF}^3_\star$-operators for the preferred semantics can be characterised by imf-mappings. The proof of this theorem is analogous to the proof of the similar theorem for propositional belief revision in [40], with two exceptions: (1) every mention of propositional logic, respectively possible worlds, is substituted by Kleene's three-valued logic, respectively three-valued interpretations, and (2) realisability of $D \star \phi$ under preferred semantics has to be accounted for and is shown to correspond to the requirement of i-modularity.

Proposition 10. Let $D = (\mathsf{At}, L, C)$ be an ADF based on a finite set of nodes At and $f : \mathfrak{D}(\mathsf{At}) \to \Omega(\mathsf{At}) \times \Omega(\mathsf{At})$ an imf-mapping for the preferred semantics. If $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ is defined by $\mathsf{prf}(D \star \psi) = \min_{\preceq_D}(\mathcal{V}(\psi))$, then $\star$ satisfies ($\mathsf{ADF}^3_\star 1$)-($\mathsf{ADF}^3_\star 6$) for prf.

Proof. We first show that $\star$ is well-defined by showing that for every $\psi \in \mathcal{L}^{\mathsf{K}}(\mathsf{At})$, there is some $D' \in \mathfrak{D}(\mathsf{At})$ s.t. $\mathsf{prf}(D') = \min_{\preceq_D}(\mathcal{V}(\psi))$. By Proposition 2 it suffices to show that $\min_{\preceq_D}(\mathcal{V}(\psi)) \neq \emptyset$ and $\min_{\preceq_D}(\mathcal{V}(\psi))$ is a $\leq_i$-antichain (i. e. for every $v, v' \in \min_{\preceq_D}(\mathcal{V}(\psi))$, $v \not\leq_i v'$). That $\min_{\preceq_D}(\mathcal{V}(\psi)) \neq \emptyset$ is easily seen by the fact that $\preceq_D$ is a total preorder on a finite set $\mathcal{V}(\mathsf{At})$. Suppose now towards a contradiction that $v, v' \in \min_{\preceq_D}(\mathcal{V}(\psi))$ and $v <_i v'$. Since $\preceq_D$ is total, $v \preceq_D v'$ or $v' \preceq_D v$. But then $v <_i v'$ contradicts $\preceq_D$ being a imf-mapping. We now show that $\star$ satisfies ($\mathsf{ADF}^3_\star 1$)-($\mathsf{ADF}^3_\star 6$):

- ($\mathsf{ADF}^3_\star 1$) is clear since by definition, $\mathsf{prf}(D \star \psi) = \min_{\preceq_D}(\mathcal{V}(\psi)) \subseteq \mathcal{V}(\psi)$.
- ($\mathsf{ADF}^3_\star 2$) Suppose $\mathsf{prf}(D) \cap \mathcal{V}(\psi) \neq \emptyset$. Since $\preceq_D$ is imf-faithful w.r.t. prf for D, $\mathsf{prf}(D) = \min_{\preceq_D}(\mathcal{V}(\mathsf{At}))$. Since $\mathcal{V}(\psi) \cap \mathsf{prf}(D) \neq \emptyset$, $\min_{\preceq_D}(\mathcal{V}(\psi)) = \min_{\preceq_D}(\mathcal{V}(\mathsf{At})) \cap \mathcal{V}(\psi) = \mathcal{V}(\psi) \cap \mathsf{prf}(D)$.
- ($\mathsf{ADF}^3_\star 3$) Suppose $\mathcal{V}(\psi) \neq \emptyset$. As $\preceq_D$ is transitive and S is finite, this implies $\min_{\preceq_D}(\mathcal{V}(\psi)) \neq \emptyset$.
- ($\mathsf{ADF}^3_\star 4$) Suppose $\mathsf{prf}(D) = \mathsf{prf}(D')$ and $\psi \equiv_{\mathsf{K}} \psi'$. Since $\preceq_D$ and $\preceq_{D'}$ are imf-faithful w.r.t. preferred, $\preceq_D = \preceq_{D'}$. Since $\psi \equiv_{\mathsf{K}} \psi'$, $\min_{\preceq_D}(\mathcal{V}(\psi)) = \min_{\preceq_D}(\mathcal{V}(\psi'))$ and thus $\mathsf{prf}(D \star \psi) = \min_{\preceq_D}(\mathcal{V}(\psi)) = \min_{\preceq_{D'}}(\mathcal{V}(\psi')) = \mathsf{prf}(D' \star \psi)$.
- ($\mathsf{ADF}^3_\star 5$) and ($\mathsf{ADF}^3_\star 6$). The case where $\mathsf{prf}(D \star \psi) \cap \mathcal{V}(\mu) = \emptyset$ is trivial.

  Suppose therefore that $\mathsf{prf}(D \star \psi) \cap \mathcal{V}(\mu) \neq \emptyset$ and suppose that $v \in \mathsf{prf}(D \star \psi) \cap \mathcal{V}(\mu)$ and suppose furthermore towards a contradiction that $v \notin \mathsf{prf}(D \star (\psi \wedge \mu)) = \min_{\preceq_D}(\mathcal{V}(\psi \wedge \mu))$, i.e. there is some $v' \in \mathcal{V}(\psi \wedge \mu)$ s.t. $v' \prec_D v$. Since $\mathcal{V}(\psi \wedge \mu) \subseteq \mathcal{V}(\psi)$, this contradicts $v \in \mathsf{prf}(D \star \psi) \cap \mathcal{V}(\mu) = \min_{\preceq_D}(\mathcal{V}(\psi)) \cap \mathcal{V}(\mu)$. Thus we have shown that $\mathsf{prf}(D \star \psi) \cap \mathcal{V}(\mu) \subseteq \mathsf{prf}(D \star (\psi \wedge \mu))$.

  Suppose now that $v \in \mathsf{prf}(D \star (\psi \wedge \mu)) = \min_{\preceq_D}(\mathcal{V}(\psi \wedge \mu))$ and suppose towards a contradiction that $v \notin \mathsf{prf}(D \star \psi) \cap \mathcal{V}(\mu)$. Notice that $v \in \mathcal{V}(\psi) \cap \mathcal{V}(\mu)$. Since we assumed $\mathsf{prf}(D \star \psi) \cap \mathcal{V}(\mu) \neq \emptyset$, there is a $v' \in \mathsf{prf}(D \star \psi) \cap \mathcal{V}(\mu)$. Since $v \in \min_{\preceq_D}(\mathcal{V}(\psi \wedge \mu))$ and we assumed $\preceq_D$ to be total, $v \preceq_D v'$. Thus, $v \in \min_{\preceq_D}(\mathcal{V}(\psi))$, contradiction.

$\square$

Proposition 11. Let $\star$ be a revision operator satisfying $(\mathsf{ADF}^3_\star 1)$-$(\mathsf{ADF}^3_\star 6)$. Then there is an imf-faithful mapping $f : \mathfrak{D}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$ (for prf) s.t. $\mathsf{prf}(D \star \psi) = \min_{f(D)}(\mathcal{V}(\psi))$.

Proof. Assume $D \in \mathfrak{D}(\mathsf{At})$. We define $f(D) = \preceq_D$ as follows: $v_1 \preceq_D v_2$ iff $v_1 \in \mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2))$. We first show that $\preceq_D$ is a total preorder:

- Totality: Consider some $v_1, v_2 \in \mathcal{V}(\mathsf{At})$. Clearly $\mathcal{V}(\mathsf{form}(v_1) \vee \mathsf{form}(v_2)) = \{v_1, v_2\}$. By $(\mathsf{ADF}^3_\star 1)$, $\mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2)) \subseteq \mathcal{V}(\mathsf{form}(v_1) \vee \mathsf{form}(v_2)) = \{v_1, v_2\}$. By $(\mathsf{ADF}^3_\star 3)$ and since $\mathcal{V}(\mathsf{form}(v_1) \vee \mathsf{form}(v_2)) \neq \emptyset$, $\mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2)) \neq \emptyset$ and thus $v_1 \in \mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2))$ or $v_2 \in \mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2))$, which implies $v_1 \preceq_D v_2$ or $v_2 \preceq_D v_1$.
- Transitivity: Suppose $v_1 \preceq_D v_2$ and $v_2 \preceq_D v_3$. We show that $v_1 \preceq_D v_3$. By $(\mathsf{ADF}^3_\star 1)$ and $(\mathsf{ADF}^3_\star 3)$ we know that $\emptyset \neq \mathsf{prf}(D \star \bigvee_{i=1}^3 \mathsf{form}(v_i)) \subseteq \{v_1, v_2, v_3\}$.
  1. Suppose first that $\mathsf{prf}(D \star \bigvee_{i=1}^3 \mathsf{form}(v_i)) \cap \{v_1, v_2\} = \emptyset$, i.e. $\mathsf{prf}(D \star \bigvee_{i=1}^3 \mathsf{form}(v_i)) = \{v_3\}$. Then by $(\mathsf{ADF}^3_\star 5)$ and $(\mathsf{ADF}^3_\star 6)$, $\mathsf{prf}\Big(D \star \Big((\bigvee_{i=1}^3 \mathsf{form}(v_i) \wedge (\mathsf{form}(v_2) \vee \mathsf{form}(v_3))\Big) = \mathsf{prf}(D \star \bigvee_{i=1}^3 \mathsf{form}(v_i)) \cap \{v_2, v_3\}$. Thus, $\mathsf{prf}(D \star \mathsf{form}(v_2) \vee \mathsf{form}(v_3)) = \{v_3\}$ and thus $v_3 \prec_D v_2$, which contradicts $v_2 \preceq_D v_3$.
  2. Suppose now that $\mathsf{prf}(D \star \bigvee_{i=1}^3 \mathsf{form}(v_i)) \cap \{v_1, v_2\} \neq \emptyset$. Since $v_1 \preceq_D v_2$, $v_1 \in \mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2))$. Using $(\mathsf{ADF}^3_\star 5)$ and $(\mathsf{ADF}^3_\star 6)$ in a similar way, we can show that $v_1 \in \mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_3))$ which implies $v_1 \preceq_D v_3$ by definition of $\preceq_D$.
- Reflexivity: By $(\mathsf{ADF}^3_\star 1)$, $\mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_1)) \subseteq \mathcal{V}(\mathsf{form}(v_1) \vee \mathsf{form}(v_1)) = \{v_1\}$ and thus (since $\mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_1)) \neq \emptyset$ with $(\mathsf{ADF}^3_\star 3)$) $\mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2) = \{v_1\}$. Thus, $v_1 \preceq_D v_1$. An analogous proof shows the cases for v_2 and v_3.

We now show that $\preceq_D$ is imf-faithful w.r.t. prf.

1. Suppose that $v_1 <_i v_2$. By $(\mathsf{ADF}^3_\star 1)$, $\emptyset \neq \mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2)) \subseteq \mathcal{V}(\mathsf{form}(v_1) \vee \mathsf{form}(v_2)) = \{v_1, v_2\}$. By Proposition 2 and since $v_1 <_i v_2$, $v_1 \notin \mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2))$ or $v_2 \notin \mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2))$ (but not both), i.e. $\mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2)) = \{v_1\}$ or $\mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2)) = \{v_2\}$ (but not both). Thus, by definition of $\preceq_D$, $v_1 \preceq_D v_2$.
2. Suppose that $v_1 \in \mathsf{prf}(D)$ and $v_2 \notin \mathsf{prf}(D)$. By $(\mathsf{ADF}^3_\star 2)$, $\mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2)) = \mathsf{prf}(D) \cap \mathcal{V}(\mathsf{form}(v_1) \vee \mathsf{form}(v_2)) = \mathsf{prf}(D) \cap \{v_1, v_2\}$. Since $v_1 \in \mathsf{prf}(D)$ and $v_2 \notin \mathsf{prf}(D)$, this means that $\mathsf{prf}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2)) = \{v_1\}$. By definition of $\preceq_D$, this implies $v_1 \prec_D v_2$ or $v_2 \prec_D v_1$.
3. Suppose that $\mathsf{prf}(D) = \mathsf{prf}(D')$. By $(\mathsf{ADF}^3_\star 4)$ it follows that $\preceq_D = \preceq_{D'}$. □

We can now show the following theorem as a corollary of Propositions 10 and 11:

Theorem 4. $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ is a trivalent ADF revision operator $\star$ for the preferred semantics prf iff there exists a function $f^\star : \mathfrak{D}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$ that is imf-faithful w.r.t. prf s.t.:

$$\mathsf{prf}(D \star \psi) = \min_{f^\star(D)}(\mathcal{V}(\psi)) \tag{6}$$

Remark 3. In the appendix, it is shown that revision of *formulas in* $\mathcal{L}^{\mathsf{K}}$, i.e. operators of the type $\star : \mathcal{L}^{\mathsf{K}} \times \mathcal{L}^{\mathsf{K}} \to \mathcal{L}^{\mathsf{K}}$, are sound and complete w.r.t. faithful total preorders over the three-valued interpretations, completely analogous to the two-valued case (see Section 2.4). Thus, revision of ADFs equals revision of three-valued formulas plus realisability.

We now show how to overcome the problems described in Example 13 by refining the naive Dalal operator based on $\preceq_D^{\mathsf{prf},\Delta}$:

Example 14. We define $\preceq_D^{\mathsf{prf},\mathsf{d+i}}$ as a lexicographic combination of the information order and the order based on distance to preferred interpretations of the ADF D under consideration. We first define the number of undecided nodes of an interpretation $v \in \mathcal{V}(\mathsf{At})$ as $\mathsf{und}(v) = |\{s \in \mathsf{At} \mid v(s) = u\}|$. We now define (given two interpretations $v_1, v_2 \in \mathcal{V}(S)$): $v_1 \preceq_D^{\mathsf{prf},\mathsf{d+i}} v_2$ iff:

1. $v_1 \in \mathsf{prf}(D)$, or
2. $v_1, v_2 \notin \mathsf{prf}(D)$ and $\mathsf{und}(v_1) < \mathsf{und}(v_2)$; or
3. $v_1, v_2 \notin \mathsf{prf}(D)$ and $\mathsf{und}(v_1) \not< \mathsf{und}(v_2)$ and $\min_{v \in \mathsf{prf}(D)}(v\Delta v_1) \leq \min_{v \in \mathsf{prf}(D)}(v\Delta v_2)$.

In the appendix we show that $\preceq^{\mathsf{prf,d+i}}$ is a total preoder.

We illustrate this preorder with the ADF D_1 from Example 3. We get the following preorder on interpretations:

$$\begin{gathered}
\top\bot\top, \quad \bot\top\top \prec_D^{\mathsf{prf},d+i} \\
\top\top\top, \quad \bot\top\bot, \quad \bot\bot\top, \quad \top\bot\bot, \quad \prec_D^{\mathsf{prf},d+i} \\
\top\top\bot, \quad \bot\bot\bot, \quad \prec_D^{\mathsf{prf},d+i} \\
\bot u\top, \quad \top u\top, \quad \bot\top u, \quad \top\bot u, \quad u\top\top, \quad u\bot\top, \quad \prec_D^{\mathsf{prf},d+i} \\
u\bot\bot, \quad \top\top u, \quad \top u\bot, \quad \bot u\bot, \quad \bot\bot u, \quad u\top\bot, \quad \prec_D^{\mathsf{prf},d+i} \\
\bot uu, \quad u\top u, \quad uu\top, \quad u\bot u, \quad \top uu, \quad \prec_D^{\mathsf{prf},d+i} \\
uu\bot \prec_D^{\mathsf{prf},d+i} \\
uuu
\end{gathered}$$

We give two examples of revisions. First consider $D \star {\sim}b$ which has as preferred models $\mathsf{prf}(D \star {\sim}c) = \{\bot\top\bot, \top\bot\bot\}$. Second, consider $D \star \odot b$ which has as preferred models $\mathsf{prf}(D \star \odot b) = \{\bot u\top, \top u\top\}$. Notice that a benefit of the approach to belief revision of ADFs presented in this section is that it is possible to revise by formulas having the third truth value u.

We now show with a second example that imf-mappings do not necessarily have to be refinements of the information-ordering on interpretations.

Example 15. We can even reverse the requirement of the second item of the definition of $\preceq_D^{\mathsf{prf,d+i}}$, i. e. prefer less informative interpretations, and still obtain an imf-mapping. We define $v_1 \preceq_D^{\mathsf{prf,d+ri}} v_2$ iff:

1. $v_1 \in \mathsf{prf}(D)$, or
2. $v_1, v_2 \notin \mathsf{prf}(D)$ and $\mathsf{und}(v_2) < \mathsf{und}(v_1)$, or
3. $v_1, v_2 \notin \mathsf{prf}(D)$ and $\mathsf{und}(v_2) \not< \mathsf{und}(v_1)$ and $\min_{v \in \mathsf{prf}(D)}(v\Delta v_1) \leq \min_{v \in \mathsf{prf}(D)}(v\Delta v_2)$.

For ADF D_1 from Example 3 we then obtain the following $\preceq_D^{\mathsf{prf,d+ri}}$-order on three-valued interpretations:

$$\begin{gathered}
\bot\top\top, \quad \top\bot\top \prec_D^{\mathsf{prfd+ri}} \\
uuu \prec_D^{\mathsf{prfd+ri}} \\
\bot uu, \quad u\top u, \quad uu\top, \quad u\bot u, \quad \top uu \prec_D^{\mathsf{prfd+ri}} \\
uu\bot \prec_D^{\mathsf{prfd+ri}} \\
\top\bot u, \quad \bot u\top, \quad \top u\top, \quad \bot\top u, \quad u\bot\top, \quad u\top\top \prec_D^{\mathsf{prfd+ri}} \\
u\top\bot, \quad \bot u\bot, \quad \bot\bot u, \quad \top\top u, \quad \top u\bot, \quad u\bot\bot \prec_D^{\mathsf{prfd+ri}} \\
\top\top\top, \quad \bot\top\bot, \quad \bot\bot\top, \quad \top\bot\bot \prec_D^{\mathsf{prfd+ri}} \\
\top\top\bot, \quad \bot\bot\bot
\end{gathered}$$

We see, for example, that $\top\bot u \prec_D^{\mathsf{prfd+ri}} u\top\bot$ since $\mathsf{und}(\top\bot u) = \mathsf{und}(u\top\bot) = 1$ and $\min_{v \in \mathsf{prf}(D)}(v\Delta\top\bot u) < \min_{v \in \mathsf{prf}(D)}(v\Delta u\top\bot)$.

To illustrate the difference with $\preceq_D^{\mathsf{prf,d+i}}$, observe that now $\mathsf{prf}(D \star {\sim}c) = \{uuu\}$.

The revision methods based on the preorders defined in the above two examples are implemented in the Java-library TweetyProject [62].[12]

[12] `http://tweetyproject.org/api/1.21/org/tweetyproject/logics/translators/adfrevision/package-summary.html`

6.4 Revision of ADFs under Grounded Semantics

In this section, we characterise revisions under the grounded semantics by a class of total preorders. The basic idea is that every "layer" contains exactly one interpretation, which ensures that every $\preceq_D$-minimal set of interpretations is singleton and thus realisable under the grounded semantics.

Definition 13. Given a semantics Sem and an ADF $D = (\mathsf{At}, L, C)$, a mapping $f : \mathfrak{D}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$ associating a total preorder $\preceq_D$ to every ADF D is an *anti-symmetric faithful mapping* (in short, asf-mapping for semantics Sem if it is faithful w.r.t. Sem and for every $D \in \mathfrak{D}(\mathsf{At})$ and for every $v_1, v_2 \in \mathcal{V}(\mathsf{At})$: if $v_1 \approx_D v_2$ then $v_1 = v_2$.

We now show that $\mathsf{ADF}^3_\star$-operator for the grounded semantics can be characterised by asf-mappings. The proof of this theorem is analogous to the proof of Theorem 4, besides that realisability of $D \star \phi$ now corresponds to the anti-symmetry condition, which means that every layer consists of a single three-valued interpretation.

Proposition 12. Let a finite set of atoms At and an asf-mapping $f : \mathfrak{D}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$ for the grounded semantics be given. If $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ is defined by $\mathsf{grounded}(D \star \psi) = \min_{\preceq_D}(\mathcal{V}(\psi))$, then $\star$ is an $\mathsf{ADF}^3_\star$-operator for $\mathsf{grounded}$.

Proof. We first show that $\star$ is well-defined by showing that for every $D \in \mathfrak{D}(\mathsf{At})$ and $\psi \in \mathcal{L}^{\mathsf{K}}$, there is some $D' \in \mathfrak{D}(\mathsf{At})$ s.t. $\mathsf{grounded}(D') = \min_{\preceq_D}(\mathcal{V}(\psi))$. By Proposition 2 it suffices to show that for every $\psi \in \mathcal{L}^{\mathsf{K}}$, $\min_{\preceq_D}(\mathcal{V}(\psi)) \neq \emptyset$ and $\min_{\preceq_D}(\mathcal{V}(\psi))$ is singleton. The former can be easily seen by the fact that $\preceq_D$ is a preorder on a finite set $\mathcal{V}(\mathsf{At})$. For the latter, suppose towards a contradiction that $\min_{\preceq_D}(\mathcal{V}(\psi)) \ni v_1, v_2$ s.t. $v_1 \neq v_2$. Since $\preceq_D$ is a total preorder, this means $v_1 \preceq_D v_2$ and $v_2 \preceq_D v_1$. But then by item 1 of Definition 13, $v_1 = v_2$, contradiction.

Showing that $\star$ satisfies ($\mathsf{ADF}^3_\star 1$)-($\mathsf{ADF}^3_\star 6$) for $\mathsf{grounded}$ is done completely analogously as in Proposition 10. $\square$

Proposition 13. Let a $\mathsf{ADF}^3_\star$-operator $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ for grounded be given. Then there is an asf-faithful mapping $f : \mathfrak{D}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$ (for $\mathsf{grounded}$) s.t. $\mathsf{grounded}(D \star \psi) = \min_{f(D)}(\mathcal{V}(\psi))$.

Proof. Assume $D \in \mathfrak{D}(\mathsf{At})$. We define $\preceq_D$ as follows: $v_1 \preceq_D v_2$ iff $v_1 \in \mathsf{grounded}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2))$. Showing that $\preceq_D$ is a total preoder that satisfies items 2-4 of Definition 13 is done completely analogously as in Proposition 11. We now show that it also satisfies item 1 of Definition 13. Indeed, suppose $v_1 \in \mathsf{grounded}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2))$. Since the grounded extension is unique, $\{v_1\} = \mathsf{grounded}(D \star \mathsf{form}(v_1) \vee \mathsf{form}(v_2))$ and thus $v_2 \preceq_D v_1$ iff $v_1 = v_2$. $\square$

We now obtain the following theorem as a corollary from Propositions 12 and 13:

Theorem 5. An operator $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ is an $\mathsf{ADF}^{3\text{'}}_\star$-operator for grounded iff there exists a function $f^\star : \mathfrak{D}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$ that is asf-faithful w.r.t. $\mathsf{grounded}$ s.t.:

$$\mathsf{grounded}(D \star \psi) = \min_{f^\star(D)}(\mathcal{V}(\psi)) \tag{7}$$

Remark 4. Kleene's logic is not the only logic for which the above characterisation result can be shown. In fact, careful inspection of the proof of Theorem 4 and 5 reveals that a similar characterisation result can be shown for any logic L for a language $\mathcal{L}^{\mathsf{L}}(\mathsf{At})$ based on an interpretation function $\sigma_{\mathsf{L}} : \mathcal{L}^{\mathsf{L}}(\mathsf{At}) \to \mathcal{V}(\mathsf{At})$ for which the following properties hold:

1. $\sigma_{\mathsf{L}}(\phi \wedge \psi) = \sigma_{\mathsf{L}}(\phi) \cap \sigma_{\mathsf{L}}(\psi)$ for any $\phi, \psi \in \mathcal{L}(\mathsf{At})$;
2. $\sigma_{\mathsf{L}}(\phi \vee \psi) = \sigma_{\mathsf{L}}(\phi) \cup \sigma_{\mathsf{L}}(\psi)$ for any $\phi, \psi \in \mathcal{L}(\mathsf{At})$;
3. for every $v \in \mathcal{V}(\mathsf{At})$ there is some $\phi_v \in \mathcal{L}^{\mathsf{L}}(\mathsf{At})$ s.t. $\sigma_{\mathsf{L}}(\phi_v) = \{v\}$.

The results for grounded semantics that will be shown in the next section can likewise be adapted to other three-valued logics. The reason we used Kleene's three-valued logic in this paper is because it satisfies the above conditions and is a well-known and -studied logic for reasoning about undecidedness.

6.5 On revising ADFs by revising acceptance conditions

The perspective on revision taken in this paper is purely semantical, in the sense that we only characterise $\mathsf{Sem}(D\star\phi)$ without specifying what exactly the acceptance conditions look like. Given the strong logical character of ADFs, in particular the essential role played by acceptance conditions in the definition of ADFs, one might hope that revision of ADFs can be somehow reduced to revision of propositional formulas. In this section, we show that it is not possible to obtain the acceptance conditions of the revised ADF $D \star \phi$ by revising the acceptance condition of the original ADF D with some propositional formula. In more detail, it is not always possible to construct, given a faithful preorder $\preceq_D$ for an ADF D, a propositional revision operator $*$ s.t. there is an ADF D' and some ϕ_s for every $s \in \mathsf{At}$ for which $C'_s = C_s * \phi_s$ and s.t. $\mathsf{Sem}(D') = \mathsf{Sem}(D \star \phi)$.

Proposition 14. Let an $\mathsf{ADF}^3_\star$-operator $\star$ be given.[13] There exist an ADF $D = (S, L, C)$ and a formula ϕ s.t. there is no propositional AGMrevision operator $*$ and a function $f : S \to \mathcal{L}$ s.t. $\mathsf{Sem}(D \star \phi) = \mathsf{Sem}(D^*_f)$ where $D^*_f = (S, L, \{C_s * f(s) \mid s \in S\})$.

Proof. Consider the ADF $D = (\{p\}, L, \{C_p = p\})$. We consider $D \star p$. For $\mathsf{Sem} \in \{\mathsf{grounded}, \mathsf{prf}, \mathsf{2mod}, \mathsf{stable}\}$, $\mathsf{Sem}(D \star p) = \{\mathsf{T}\}$ by $\mathsf{ADF}^2_\star 1$ respectively $\mathsf{ADF}^3_\star 1$. Suppose now there is a propositional revision operator $*$ and some $\phi \in \mathcal{L}$ s.t. $D' = (\{p\}, L, \{C_p * \phi\})$ and $\mathsf{Sem}(D \star p) = \{\mathsf{T}\}$. This is possible only if $C_p * \phi_p \equiv \top$.

With R2, $C_p * \phi = C_p \wedge \phi$ or C_p inconsistent with ϕ_p. Notice that $C_p \wedge \phi_p \equiv \top$ is impossible. Thus, C_p inconsistent with ϕ. Notice that $C_p * \phi \equiv \top$ iff $\phi \equiv \top$ with R1-R3. However, C_p cannot be inconsistent with ϕ if $\phi \equiv \top$. □

The critical reader might remark that the example above can be avoided if one chooses to contract by a formula ϕ instead of revising by it. However, a similar example that shows that contraction is not always sufficient can be constructed. Combining contraction and revision is a feasible option, which falls outside the scope of this paper.

7 The role of equivalence in belief revision

Equivalence for non-monotonic formalisms has received considerable attention, as so-called "classical" equivalence, i. e. two ADFs having the same Sem-interpretations, might not always capture the intuition of equivalence of two ADFs sufficiently, as two classically equivalent ADFs might turn out not to be classically equivalent any more when the same information is added to both ADFs:

Example 16. Let $D_1 = (\{a, b\}, L_1, C^1_a = \neg a, C^1_b = a)$ and $D_2 = (\{a, b\}, L_1, C^2_a = \neg a, C^2_b = \neg a)$. Notice that $\mathsf{prf}(D_1) = \mathsf{prf}(D_2) = v$ where $v(a) = v(b) = u$. However, if we add the argument c that attacks a, we obtain the following ADFs: $D'_1 = (\{a, b, c\}, L_1, C'^1_a = \neg a \wedge \neg c, C'^1_b = a, C'^1_c = \top)$ and $D'_2 = (\{a, b, c\}, L_1, C'^2_a = \neg a \wedge \neg c, C'^2_b = \neg a, C'^2_c = \top)$. We see that $\mathsf{prf}(D'_1) = \{v_1\}$ and $\mathsf{prf}(D'_2) = \{v_2\}$ where: $v_1(a) = v_1(b) = \mathsf{F}$ and $v_1(c) = \mathsf{T}$ whereas $v_2(a) = \mathsf{F}$ and $v_2(b) = v_2(c) = \mathsf{T}$.

Strong equivalence is a stronger notion of equivalence that formalises exactly the intuition that two ADFs are equivalent if and only if they have the same extensions after the addition of any additional information. Strong equivalence for ADFs has been defined in [39] and characterised for the admissible, complete, preferred and grounded semantics. We recall the definition and results here.

For many formalisms, addition of knowledge can be modelled using set-theoretic union. For ADFs, this is not feasible for several reasons. Firstly, combining two ADFs under set-theoretic union does not result in a new ADF but rather in a set of ADFs. Secondly, one has to ensure that one models appropriately the combination of two ADFs with shared atoms. Consider e. g. two ADFs $D_1 = (\{a\}, L_1, C^1_a)$ and $D_2 = (\{a\}, L_2, C^2_a)$ with $C^1_a = a$ and $C^2_a = \neg a$. Clearly, the combination of ADFs has to be modelled on the basis of some logical operator combining C^1_a and C^2_a in a single new condition C_a. We specify a general model of addition of ADFs which allows for the combination of conditions using either disjunction or conjunction. Given a set of atoms At, an *and-or-assignment for* At is a mapping $\odot : \mathsf{At} \to \{\wedge, \vee\}$. Intuitively, an and-or-assignment specifies for every atom $s \in \mathsf{At}$ whether conditions for s will be combined using $\wedge$ or using $\vee$. We now define the combination of two ADFs:

[13] Recall that $\mathsf{ADF}^3_\star$-operator are defined in Definition 10.

Definition 14 ([39])**.** Let $D_1 = (\mathsf{At}_1, L_1, C_1)$ and $D_2 = (\mathsf{At}_2, L_2, C_2)$ be two ADFs and $\odot$ an and-or-assignment for At. Define $D_1 \Cup_\odot D_2 = (\mathsf{At}_1 \cup \mathsf{At}_2, L_1 \cup L_2, C^\odot)$ with and $C^\odot = \{C^\odot_s\}_{s\in\mathsf{At}}$, where:[14]

$$C^\odot_s = \begin{cases} C^1_s \odot(s) C^2_s & \text{if } s \in \mathsf{At}_1 \cap \mathsf{At}_2 \\ C^1_s & \text{if } s \in \mathsf{At}_1 \setminus \mathsf{At}_2 \\ C^2_s & \text{if } s \in \mathsf{At}_2 \setminus \mathsf{At}_1 \end{cases}$$

Example 17. Consider D as in Example 3, $D' = (\{a, b, d\}, L', C)$ with $C_a = b$, $C_b = d \wedge \neg a$ and $C_d = \neg a$, and $\odot(a) = \odot(b) = \wedge$ and $\odot(c) = \odot(d) = \vee$. Then $D_1 \Cup_\odot D_2 = (\{a, b, c, d\}, L_1 \cup L_2, C^\odot)$ where: $C^\odot_a = \neg b \wedge b$, $C^\odot_b = \neg a \wedge d \wedge \neg a$, $C^\odot_c = \neg a \vee \neg b$ and $C^\odot_d = \neg a$.

We now define strong equivalence for ADFs as follows:

Definition 15 ([39])**.** Two ADFs $D_1 = (\mathsf{At}, L_1, C_1)$ and $D_2 = (\mathsf{At}, L_2, C_2)$ are strongly equivalent under semantics Sem iff for any $D \in \mathfrak{D}(\mathsf{At})$ and any and-or-assignment $\odot$ for At, $\mathsf{Sem}(D_1 \Cup_\odot D) = \mathsf{Sem}(D_2 \Cup_\odot D)$.

The notion of equivalence is a concept which is used in the definition and characterisation of revision. In more detail, on the syntactic side, it is used in the axiom $\mathsf{ADF}^3_\star 4$ by requiring that two equivalent ADFs, when revised by the same formula, result in an equivalent revised ADF. This is reflected on the semantic side by the requirement that equivalent ADFs give rise to the same total preorders on faithful mappings (and their specialisations). In some approaches to revision of logic programs [16], the notion of strong equivalence has been used. Even though we have based our approach on the notion of classical equivalence, it is not hard to adapt our approach to use the notion of strong equivalence instead. In more detail, the postulates from Definition 10 and the corresponding representation results can be adapted as follows:

Definition 16. An operator $\star$ is a *trivalent ADF SE-revision operator* (in short, $\mathsf{ADF}^3_\star$-SE-operator) for a semantics Sem iff $\star$ satisfies (for any $\phi, \psi, \mu \in \mathcal{L}^\mathsf{K}$) ($\mathsf{ADF}^3_\star 1$),($\mathsf{ADF}^3_\star 2$), ($\mathsf{ADF}^3_\star 3$),($\mathsf{ADF}^3_\star 5$)($\mathsf{ADF}^3_\star 6$) and:

($\mathsf{ADF}^3_\star 4$') If D and D' are strongly equivalent under Sem and $\psi \equiv_\mathsf{K} \psi'$ then $D \star \psi$ and $D' \star \psi'$ are strongly equivalent under Sem.

For the semantic characterisation of trivalent ADF SE-revision operators, we simply need to adopt the definition of faithful mappings as follows:

Definition 17. Given a semantics Sem and an ADF $D = (\mathsf{At}, L, C)$, a mapping $f : D \mapsto \preceq_D$ associating a total preorder $\preceq_D$ to every ADF D is an *SE-faithful mapping* for semantics Sem if, for every $D \in \mathfrak{D}(\mathsf{At})$ and for every $v_1, v_2 \in \mathcal{V}(\mathsf{At})$:

1. if $v_1 \in \mathsf{Sem}(D)$ then $v_1 \preceq_D v_2$; and
2. if $v_1 \in \mathsf{Sem}(D)$ and $v_2 \notin \mathsf{Sem}(D)$ then $v_1 \prec_D v_2$; and
3. if D and D' are strongly equivalent under Sem, then $\preceq_D = \preceq_{D'}$.

Notice that any SE-faithful mapping is faithful, but not vice-versa.

imf-SE-faithful, respectively asf-SE-faithful, mappings are defined exactly as imf-, respectively asf-faithful, mappings, only that they are required to be SE-faithful instead of faithful.

We now provide characterisation results analogous to the ones for $\mathsf{ADF}^3_\star$-revision operators:

Theorem 6. An operator $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}^\mathsf{K}(\mathsf{At}) \to \mathcal{L}^\mathsf{K}(\mathsf{At})$ is:

1. an $\mathsf{ADF}^3_\star$'-SE-operator for preferred semantics iff there exists a function $f^\star : \mathfrak{D}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$ that is imf-SE-faithful w.r.t. prf s.t.:
$$\mathsf{prf}(D \star \psi) = \min_{f^\star(D)}(\mathcal{V}(\psi)) \tag{8}$$
2. an $\mathsf{ADF}^3_\star$'-SE-operator for grounded semantics iff there exists a function $f^\star : \mathfrak{D}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$ that is asf-SE-faithful w.r.t. grounded s.t.:
$$\mathsf{grounded}(D \star \psi) = \min_{f^\star(D)}(\mathcal{V}(\psi)) \tag{9}$$

[14] This notion of composition of ADFs is a generalisation of that of [27].

Proof. The proof is entirely analogous to the proof of Theorems 4 and 5, with the exception of the proof of the equivalence of $\mathsf{ADF}^3_\star 4$' and the third condition of SE-faithfulness. However, it is easy to see that the proof of the third condition of faithfulness using $\mathsf{ADF}^3_\star 4$ respectively the proof of $\mathsf{ADF}^3_\star 4$ from the third condition of faithfulness in Theorems 4 and 5 can be easily transformed into a proof of the respective adapted condition. ☐

We illustrate the difference between $\mathsf{ADF}^3_\star$-revision and $\mathsf{ADF}^3_\star$'-revision with a continuation of Example 16.

Example 18. Consider again the ADFs D_1 and D_2 from Example 16. The following preorders $\preceq_1$ and $\preceq_2$ over $\mathcal{V}(\{a,b\})$ are SE-faithful but not faithful:

$$uu \prec_1 \mathsf{FF}, \mathsf{FT} \prec_1 \mathsf{FF} \prec_1 \mathsf{TT} \prec_1 u\mathsf{F}, \mathsf{F}u, \mathsf{T}u, u\mathsf{T}$$

$$uu \prec_2 \mathsf{FF} \prec_2 \mathsf{FF} \prec_2 \mathsf{FT} \prec_2 \mathsf{TT} \prec_2 u\mathsf{F}, \mathsf{F}u, \mathsf{T}u, u\mathsf{T}$$

When revising D_1 repsectively D_2 with $\neg a$, we see that $\mathsf{prf}(D_1 \star \neg a) = \{\mathsf{FF}, \mathsf{FT}\}$ whereas $\mathsf{prf}(D_2 \star \neg a) = \{\mathsf{FF}\}$. This revision operator is not a $\mathsf{ADF}^3_\star$-revision operator as $\mathsf{prf}(D_1) = \mathsf{prf}(D_2)$ yet $\mathsf{prf}(D_1 \star \neg a) = \mathsf{prf}(D_2 \star \neg a) = \{\mathsf{FF}\}$ and thus $\mathsf{ADF}^3_\star 4$ is violated.

8 Revision of ADFs under Possibilistic logic

In the developments in Sections 4-6 above, we have used Kleene's logic as a logic underlying revision, and playing e. g. a role in the formulation of postulates for revision operators. As pointed out in Remark 4, this choice bears some flexibility, as other logics that satisfy the conditions stipulated in that remark can also be used. As in [39] it was shown that possibilistic logic underlies abstract dialectical argumentation, in the sense that the evaluations of formulas under possibilistic logic coincide with $\sqcap_i [v]^2(\phi)$ for any formula $\phi \in \mathcal{L}$, we investigate here the question whether possibilistic logic can also be used as a logic underlying revision of ADFs.

We first recall the necessary preliminaries on possibilistic logic. In [22], a three-valued logic inspired by possibility theory [21] is defined. The basic idea behind this logic is to base lower and upper bounds of the evaluation of a formula using a *possibility* and a *necessity measure*. In more detail, given a three-valued interpretation v over At (see Section 2.5), the set of two-valued interpretations extending a valuation v is defined as $[v]^2 = \{w \in \Omega(\mathsf{At}) \mid v \leq_i w\}$.[15]

Definition 18. Given $v \in \mathcal{V}(\mathsf{At})$, the *necessity measure* $\mathcal{N}_v$ and the *possibility measure* Π_v based on v are functions : $\mathcal{N}_v : \mathcal{L}(\mathsf{At}) \to \{\mathsf{T}, \mathsf{F}\}$ and $\Pi_v : \mathcal{L}(\mathsf{At}) \to \{\mathsf{T}, \mathsf{F}\}$

$$\Pi_v(\phi) = \begin{cases} \mathsf{T} & \text{iff } \omega \models \phi \text{ for some } \omega \in [v]^2 \\ \mathsf{F} & \text{otherwise} \end{cases}$$

$$\mathcal{N}_v(\phi) = \begin{cases} \mathsf{T} & \text{iff } \omega \models \phi \text{ for every } \omega \in [v]^2 \\ \mathsf{F} & \text{otherwise} \end{cases}$$

We can now derive a three-valued evaluation $v^{\mathsf{poss}} : \mathcal{L}(\mathsf{At}) \to \{\mathsf{T}, \mathsf{F}, u\}$ by stating that:[16]

$$v^{\mathsf{poss}}(\phi) = \begin{cases} \mathsf{T} & \text{iff } \mathcal{N}_v(\phi) = \mathsf{T} \\ u & \text{iff } \mathcal{N}_v(\phi) = \mathsf{F} \text{ and } \Pi_v(\phi) = \mathsf{T} \\ \mathsf{F} & \text{iff } \mathcal{N}_v(\phi) = \Pi_v(\phi) = \mathsf{F} \end{cases}$$

Thus, $v^{\mathsf{poss}}(\phi) = \mathsf{T}[\mathsf{F}]$ means that ϕ is necessary true[false] (i. e. $\mathcal{N}_v(\phi) = \Pi_v(\phi) = \mathsf{T}[\mathsf{F}]$) whereas $v^{\mathsf{poss}}(\phi) = u$ means that ϕ is possible ($\Pi_v(\phi) = \mathsf{T}$) but not necessary ($\Pi_v(\phi) = \mathsf{F}$). We denote with $\mathcal{V}^{\mathsf{poss}}(\phi)$ the set of interpretations that satisfy ϕ according to possibilistic logic, i. e. $\mathcal{V}^{\mathsf{poss}}(\phi) = \{v \in \mathcal{V} \mid v^{\mathsf{poss}}(\phi) = \mathsf{T}\}$.

We notice that possilistic logic is not truth-functional:

[15] In [10], instead of two-valued interpretations extending a valuation, the notion of *epistemic set* E_v is used, which defined as: $E_v = \{v' \in \Omega \mid v \leq_i v'\}$. It is clear that $E_v = [v]^2$ for any $v \in \mathcal{V}$.

[16] Notice that this enumeration of cases is exhaustive, as for any $v \in \mathcal{V}(\mathsf{At})$ and any $\phi \in \mathcal{L}(\mathsf{At})$, $\mathcal{N}_v(\phi) \leq_{\mathsf{T}} \Pi_v(\phi)$.

Example 19. Consider the interpretation v over $\{a, b\}$ with $v(a) = v(b) = u$. Notice that $\mathcal{N}_v(a \vee \neg a) = \mathsf{T}$ and thus $v^{\mathsf{poss}}(a \vee \neg a) = \mathsf{T}$. However, $\mathcal{N}_v(a \vee b) = \mathcal{N}_v(\neg a) = \mathsf{F}$ and $\Pi_v(a \vee b) = \Pi_v(\neg a) = \mathsf{T}$. Thus, even though $v(a) = v^{\mathsf{poss}}(\neg a) = v(b) = u$, $v^{\mathsf{poss}}(a \vee b) \neq v^{\mathsf{poss}}(a \vee \neg a)$.

We now answer negatively the question as to whether revision under possibilistic logic is characterisable as selections of a faithful ranking over $\mathcal{V}$. In more detail, we will show that there are several problems with modelling revision under possibilistic logic. For this, we restrict attention to revision by formulas in the classical language, as there is no straightforward way to express undecidedness in possibilistic logic as defined in [22].

Definition 19. An operator $\star$ is a *possibilistic ADF revision operator* (in short, possADFR-operator) for a semantics Sem iff $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ satisfies (for any $\phi, \psi, \mu \in \mathcal{L}$)[17]:

(possADFR1) $\mathsf{Sem}(D \star \psi) \subseteq \mathcal{V}^{\mathsf{poss}}(\psi)$.
(possADFR2) If $\mathsf{Sem}(D) \cap \mathcal{V}^{\mathsf{poss}}(\psi) \neq \emptyset$ then $\mathsf{Sem}(D \star \psi) = \mathsf{Sem}(D) \cap \mathcal{V}^{\mathsf{poss}}(\psi)$.
(possADFR3) If $\mathcal{V}^{\mathsf{poss}}(\psi) \neq \emptyset$ then $\mathsf{Sem}(D \star \psi) \neq \emptyset$.
(possADFR4) If $\mathsf{Sem}(D) = \mathsf{Sem}(D')$ and $\mathcal{V}^{\mathsf{poss}}(\psi) = \mathcal{V}^{\mathsf{poss}}(\psi')$ then $\mathsf{Sem}(D \star \psi) = \mathsf{Sem}(D' \star \psi')$.
(possADFR5) $\mathsf{Sem}(D \star \psi) \cap \mathcal{V}^{\mathsf{poss}}(\mu) \subseteq \mathsf{Sem}(D \star (\psi \wedge \mu))$.
(possADFR6) If $\mathsf{Sem}(D \star \psi) \cap \mathcal{V}^{\mathsf{poss}}(\mu) \neq \emptyset$, then $\mathsf{Sem}(D \star (\psi \wedge \mu)) \subseteq \mathsf{Sem}(D \star \psi) \cap \mathcal{V}^{\mathsf{poss}}(\mu)$.

An example of such an operator can be obtained by considering the ADF $D = (\{a, b\}, L, C)$ with $C_a = \top$ and $C_b = a$ and consider the total preorder (over the alphabet $\{a, b\}$):

$$\mathsf{TT} \prec uu \prec uu \prec u\mathsf{F} \prec u\mathsf{T} \prec \mathsf{F}u \prec \mathsf{T}u \prec \mathsf{FT} \prec \mathsf{TF} \prec \mathsf{FF}$$

We define again revision (under the preferred semantics) by means of the equation

$$\mathsf{prf}(D \star \phi) = \min_{\preceq} \mathcal{V}^{\mathsf{poss}}(\phi)$$

It can be checked that this revision operator satisfies all the possADFR-postulates. As an example of a concrete revision, consider revision by $\neg b$, which results in $\mathsf{prf}(D \star \neg b) = \{\bot\bot\}$ as $\mathcal{V}^{\mathsf{poss}}(\neg b) = \{\mathsf{FF}, u\mathsf{F}, \mathsf{TF}\}$ and FF is the $\preceq$-minimal interpretation.

The first problem is that not every revision that satisfies the postulates possADFR1-possADFR6 can be modelled by means of a total preorder. The way to prove this is similar to the proof of an analogous result for revision of Horn-theories from [17].

Proposition 15. There exist revision operators $\star$ that satisfy all postulates from Definition 19 yet for which there exists no total preorder $\preceq$ over $\mathcal{V}(\mathsf{At})$ s.t. $\min_{\preceq}(\mathcal{V}^{\mathsf{poss}}(\phi)) = \mathsf{Sem}(D \star \phi)$ (for $\mathsf{Sem} \in \{\mathsf{grounded}, \mathsf{prf}\}$).

Proof. Consider the relation $uu \prec \mathsf{TT}$, $\mathsf{TT} \prec \mathsf{TF}$, $\mathsf{TF} \prec \mathsf{F}u$ and $v_1 \prec v_2$ for every $v_1 \in \{uu, \mathsf{TT}, \mathsf{TF}, \mathsf{F}u\}$ and $v_2 \in \mathcal{V}(\{a, b\}) \setminus \{uu, \mathsf{TT}, \mathsf{TF}, \mathsf{F}u\}$. Furthermore, we impose an arbitrary anti-symmetric order over $\mathcal{V}(\{a, b\}) \setminus \{uu, \mathsf{TT}, \mathsf{TF}, \mathsf{F}u\}$, i.e. for every $v_1, v_2 \in \mathcal{V}(\mathsf{At}) \setminus \{uu, \mathsf{TT}, \mathsf{TF}, \mathsf{F}u\}$, $v_1 \prec v_2$ or $v_2 \prec v_1$. This ensures realisability under prf and grounded semantics.

We first show the following two lemmas:

Lemma 2. There are some formulas ϕ_1, ϕ_2, ϕ_3 s.t.

- $\mathsf{TT}, \mathsf{TF} \in \mathcal{V}^{\mathsf{poss}}(\phi_1)$ and $\mathsf{F}u, uu \notin \mathcal{V}^{\mathsf{poss}}(\phi_1)$,
- $\mathsf{TF}, \mathsf{F}u \in \mathcal{V}^{\mathsf{poss}}(\phi_2)$ and $\mathsf{TT}, uu \notin \mathcal{V}^{\mathsf{poss}}(\phi_2)$, and
- $\mathsf{TT}, \mathsf{F}u \in \mathcal{V}^{\mathsf{poss}}(\phi_3)$ and $\mathsf{TF}, uu \notin \mathcal{V}^{\mathsf{poss}}(\phi_3)$,

Proof. This can be seen by setting $\phi_1 = p$, $\phi_2 = p\overline{q} \vee \overline{p}$ and $\phi_3 = \overline{p} \vee pq$. ☐

Lemma 3. For any formula $\phi \in \mathcal{L}(\mathsf{At})$, if $\mathsf{TT}, \mathsf{TF}, \mathsf{F}u \in \mathcal{V}^{\mathsf{poss}}(\phi)$ then also $uu \in \mathcal{V}^{\mathsf{poss}}(\phi)$.

Proof. Suppose $\mathsf{TT}, \mathsf{TF}, \mathsf{F}u \in \mathcal{V}^{\mathsf{poss}}(\phi)$. Since $\mathsf{F}u \in \mathcal{V}^{\mathsf{poss}}(\phi)$, $\omega \models \phi$ for every $\omega \in [\mathsf{F}u]^2$. Likewise, for every $\omega \in [\mathsf{TT}]^2 \cup [\mathsf{F}u]^2$, $\omega \models \phi$. Since $[\mathsf{F}u]^2 \cup [\mathsf{TT}]^2 \cup [\mathsf{TF}]^2 = \Omega(\mathsf{At})$ (which can be seen by observing that $\omega \in [\mathsf{F}u]^2 \cup [\mathsf{TT}]^2 \cup [\mathsf{TF}]^2$ iff $\omega \models \neg p$ or $\omega \models p \wedge q$ or $\omega \models p \wedge \neg q$), also $\omega \models \phi$ for ever $\omega \in [uu]^2$. ☐

[17] Recall that possibilistic logic is defined over a classical propositional language.

We now show (1): $\star$ defined by $\min_{\preceq}(\mathcal{V}^{\mathsf{poss}}(\phi)) = \mathsf{Sem}(D \star \phi)$ satisfies all postulates possADFR1-possADFR6.

possADFR**1** immediate since $\mathsf{Sem}(D \star \phi) = \min_{\preceq}(\mathcal{V}^{\mathsf{poss}}(\phi))$.

possADFR**2** This is immediate, since $\mathsf{Sem}(D) = \{uu\}$.

possADFR**3** Suppose $\mathcal{V}^{\mathsf{poss}}(\psi) \neq \emptyset$. If $\mathcal{V}^{\mathsf{poss}}(\psi) \neq \emptyset$ or $\mathcal{V}^{\mathsf{poss}}(\psi) \cap \{\mathsf{TT}, \mathsf{TF}, \mathsf{F}u\} = \emptyset$ then $\min_{\preceq}(\mathcal{V}^{\mathsf{poss}}(\psi))$ is well-defined and thus non-empty. Suppose now $uu \notin \mathcal{V}^{\mathsf{poss}}(\psi)$ and $\mathcal{V}^{\mathsf{poss}}(\psi) \cap \{\mathsf{TT}, \mathsf{TF}, \mathsf{F}u\} \neq \emptyset$. We show the case for $\mathsf{TT} \in \mathcal{V}^{\mathsf{poss}}(\phi)$, the other cases are similar. With Lemma 3, $\mathsf{TF} \notin \mathcal{V}^{\mathsf{poss}}(\psi)$ or $\mathsf{F}u \notin \mathcal{V}^{\mathsf{poss}}(\psi)$. If neither are the case, $\min_{\preceq}(\mathcal{V}^{\mathsf{poss}}(\psi)) = \{\mathsf{TT}\}$ and we are done. Assume now $\mathsf{TF} \notin \mathcal{V}^{\mathsf{poss}}(\psi)$ and $\mathsf{F}u \in \mathcal{V}^{\mathsf{poss}}(\psi)$. Then $\min_{\preceq}(\mathcal{V}^{\mathsf{poss}}(\psi)) = \min_{\preceq}(\{\mathsf{TT}, \mathsf{F}u\}) = \{\mathsf{TF}\}$ and thus we are done. The other case is similar.

possADFR**4** Immediate since if $\mathcal{V}^{\mathsf{poss}}(\psi) = \mathcal{V}^{\mathsf{poss}}(\phi)$, the corresponding $\preceq$-minimal selections will be identical as well.

possADFR**5 and** possADFR**6** Here we show that $\mathsf{Sem}(D \star \psi) \cap \mathcal{V}^{\mathsf{poss}}(\mu) \neq \emptyset$ implies $\mathsf{Sem}(D \star \psi) \cap \mathcal{V}^{\mathsf{poss}}(\mu) = \mathsf{Sem}(D \star (\psi \wedge \mu)$. This is done similarly as in e. g. the proof of Proposition 10, with the only additional task of showing that $\mathsf{Sem}(D \star (\psi \wedge \mu) \neq \emptyset$. The only non-trivial case is the case where $\mathsf{Sem}(D \star \psi) \cap \mathcal{V}^{\mathsf{poss}}(\mu))$ contains two out of three interpretations from among $\mathsf{TT}, \mathsf{TF}, \mathsf{F}u$. Assume thus that $\mathsf{Sem}(D \star \psi) \cap \mathcal{V}^{\mathsf{poss}}(\mu) \supseteq \{\mathsf{TT}, \mathsf{TF}\}$. If $\mathsf{F}u \in \mathsf{Sem}(D \star \psi) \cap \mathcal{V}^{\mathsf{poss}}(\mu)$, with Lemma 3, $uu \in \mathsf{Sem}(D \star \psi) \cap \mathcal{V}^{\mathsf{poss}}(\mu)$ and thus $\min_{\preceq}(\mathsf{Sem}(D \star \psi)) \cap \mathcal{V}^{\mathsf{poss}}(\mu) = \{uu\}$. Assume thus that $\mathsf{F}u \notin \mathsf{Sem}(D \star \psi) \cap \mathcal{V}^{\mathsf{poss}}(\mu)$. Then $\min_{\preceq}(\mathsf{Sem}(D \star \psi)) \cap \mathcal{V}^{\mathsf{poss}}(\mu) = \{\mathsf{TT}\}$. The other cases are similar.

We now show (2): there is no total preorder $\preceq'$ s.t. $\min_{\preceq}(\mathcal{V}^{\mathsf{poss}}(\phi)) = \min_{\preceq'}(\mathcal{V}^{\mathsf{poss}}(\phi))$ for every $\phi \in \mathcal{L}$. Suppose towards a contradiction that there exists such a total preorder $\preceq'$. With Lemma 2, $\mathsf{TT}, \mathsf{TF} \in \mathcal{V}^{\mathsf{poss}}(p)$ and $\mathsf{F}u, uu \notin \mathcal{V}^{\mathsf{poss}}(p)$. Thus, $\min_{\preceq'} \mathcal{V}(p) = \{\mathsf{TT}\}$ which implies $\mathsf{TT} \prec' \mathsf{TF}$. Likewise, we can establish $\mathsf{TF} \prec' \mathsf{F}u$ and $\mathsf{F}u \prec' \mathsf{TT}$. But then with transitivity of $\prec'$, $\mathsf{TT} \prec' \mathsf{TT}$, contradiction.

The proposition follows immediately from (1) and (2). □

A second problem, again analogous to a problem identified for revision of Horn theories in [17], is that some total preorders cannot be distinguished. In more detail, distinct total preorders can induce an identical revision operator $\star$. For example, the following total preorders give rise to identical revision:

1. The preorder $\preceq_1$ with as first three levels $uu \prec_1 \mathsf{T}u \prec_1 \mathsf{F}u$ and as fourth level containing all other interpretations.
2. The preorder $\preceq_2$ with as first three levels $uu \prec_2 \mathsf{F}u \prec_2 \mathsf{T}u$ and as fourth level containing all other interpretations.
3. The preorder $\preceq_3$ with as first levels $uu \prec_3 \mathsf{T}u, \mathsf{F}u$ and as third level containing all other interpretations.

We can easily show that there exists no possADFR-operators $\star_1$, $\star_2$ and $\star_3$ s.t. $\mathsf{Sem}(D \star_i \phi) = \min_{\preceq_i}(\phi)$ for any $i = 1, 2, 3$ and any $\phi \in \mathcal{L}$ and such that $\star_i \neq \star_j$ for $i \neq j$ and $i, j = 1, 2, 3$. To see that there is no possADFR-operator that distinguishes these three preorders, suppose towards a contradiction that there is some ϕ s.t. $\min_{\preceq_i}(\mathcal{V}(\phi)) \neq \min_{\preceq_j}(\mathcal{V}(\phi))$ for some $i, j = 1, 2, 3$ and $i \neq j$. This would mean that $\mathcal{V}^{\mathsf{poss}}(\phi) \ni \mathsf{T}u, \mathsf{F}u$ and $uu \notin \mathcal{V}^{\mathsf{poss}}(\phi)$. But this is impossible, since $\mathsf{T}u, \mathsf{F}u \in \mathcal{V}^{\mathsf{poss}}(\phi)$ implies $uu \in \mathcal{V}^{\mathsf{poss}}(\phi)$ (since $[\mathsf{T}u]^2 \cup [\mathsf{F}u]^2 = [uu]^2$).

These problems can be solved by solutions analogous to the one given for revision of Horn theories in [17], i. e. by requiring additional postulates that ensure acyclicity of the total preorders used to semantically represent revision operators. However, working out the full details of such a solution falls outside the scope of this paper.

9 Nonmonotonic inference and defeasible conditionals for ADFs

In this section, we study interrelations between $\mathsf{ADF}^3_\star$-operators, trivalent defeasible conditionals, and nonmonotonic inference based on three-valued logic. We first define nonmonotonic inference based on three-valued logic and show how they can be equivalently viewed as the acceptance of trivalent defeasible conditionals. Thereafter, we define both *static conditionals* and *dynamic conditionals* for ADFs, which are defined using the Ramsey test on the basis of the revision operators developed and studied above. Finally, we show that the interrelations between revision, conditionals, and inference relations known from propositional beliefs hold also in our argumentative setting.

9.1 Three-valued nonmonotonic inference and defeasible conditionals

Like in the two-valued case, we will investigate non-monotonic inferences $\phi \mathrel{|\!\sim} \psi$, read as "if ϕ then typically ψ", over $\mathcal{L}^{\mathsf{K}}$ on the basis of total preorders over $\mathcal{V}(\mathsf{At})$. Nonmonotonic inference on the basis of three-valued logics such as K can be defined completely analogously to the two-valued case, by specifying total preorders $\preceq$ that express a comparative measure of plausibility over the set of three-valued interpretations. We can then easily generalise the definition of conditional inference to sets of three-valued interpretations. Given a set of atoms At, we assume a total preorder $\preceq$ over $\mathcal{V}(\mathsf{At})$. We can now define conditional inference based on Kleene's three-valued logics as follows:

Definition 20. Given a set of atoms At, a total preorder $\preceq$ over $\mathcal{V}(\mathsf{At})$, and some $\phi, \psi \in \mathcal{L}^{\mathsf{K}}(\mathsf{At})$, $\phi \mathrel{|\!\sim}^{\mathsf{K}}_{\preceq} \psi$ iff $v \prec v'$ for some $v \in \min_{\preceq}(\mathcal{V}(\phi \wedge \psi))$ and $v' \in \min_{\preceq}(\mathcal{V}(\phi \wedge {\sim}\psi))$.[18]

It can be noticed that this is just a special case of what [43, 49] call a *preferential model*. Notice the choice of negation in the definition above. This is to ensure that an inference $\phi \mathrel{|\!\sim}^{\mathsf{K}}_{\preceq} \psi$ is valid iff all $\preceq$-minimal worlds that validate ϕ also validate ψ , in accordance with [43]. This is ensured by using $\sim$, as $v({\sim}\psi) = \top$ iff $v(\psi) \in \{\bot, u\}$, i. e. if ψ not explicitly true. If we would have used $\neg$ in the above definition, (minimal) interpretations which make ϕ true and ψ undecided could still be preferred over (minimal) interpretations that make ϕ and ψ true. The following fact shows that indeed, ϕ typically entails ψ relative to $\preceq$ if all typical ϕ-interpretations (according to $\preceq$) entail ψ:

Fact 5. $\phi \mathrel{|\!\sim}^{\mathsf{K}}_{\preceq} \psi$ iff $\min_{\preceq} \mathcal{V}(\phi) \subseteq \mathcal{V}(\psi)$.

This fact shows that, just like in the case of classical nonmonotonic inference relations, three-valued nonmonotonic inference relations obtained on the basis of a total preorder can be equivalently viewed as conditional inference relations on the basis of the selection function $\min_{\preceq}$. In other words, conditionals $(\psi|\phi)$, defined on the basis of the selection function $\min_{\preceq}$ can be simply seen as the syntactic counterparts of the nonmonotonic inference relation $\mathrel{|\!\sim}^{\mathsf{K}}_{\preceq}$.

We illustrate the definition of conditional inference based on Kleene's three-valued logic as follows:

Example 20. Consider the preorder $\preceq^{\mathsf{prf},d+i}_{D}$ from Example 14. For ease of notation we set $\preceq = \preceq^{\mathsf{prf},d+i}_{D}$. We see that e. g. $\top \mathrel{|\!\sim}_{\preceq} c$ as $\min_{\preceq}(\mathcal{V}(\top) \subseteq \mathcal{V}(c)$. Likewise, $\odot b \mathrel{|\!\sim}_{\preceq} c$ as the $\preceq$-minimal interpretations validating $\odot b$ (i. e. the $\preceq$-minimal interpretations v with $v(b) = u$) all set c to true.

We show that any inference relation based on a total preorder over $\mathcal{V}(\mathsf{At})$ satisfies (REF), (CUT), (CM), (RW), (LLE), (OR) (see Section 2.3) and a postulate we call *weak Rational Monotony* (wRM):

(wRM) $\phi \mathrel{|\!\sim} \gamma$ and $\phi \mathrel{|\!\not\sim} {\sim}\psi$ implies $\phi \wedge \psi \mathrel{|\!\sim} \gamma$

In the context of three-valued logics, the difference between (RM) and (wRM) is the following: the antecendent of (wRM) requires that from ϕ, neither $\odot\psi$ nor $\neg\psi$ can be derived, i. e. if ϕ then normally ψ is neither false nor undecided. (RM), on the other hand, has a weaker antecedent, namely that $\neg\psi$ cannot be derived, i. e. normally ψ is not false if ϕ is accepted.

Analogously to Proposition 1, nonmonotonic inference relations induced by total preorders over three-valued interpretation satisfies all the KLM-postulates (as presented in Section 2.3) as well as the non-Horn postulate (wRM):

Proposition 16. Given a set of atoms At and a total preorder $\preceq$ over $\mathcal{V}(\mathsf{At})$, $\mathrel{|\!\sim}^{\mathsf{K}}_{\preceq}$ satisfies (REF), (CUT), (CM), (RW), (LLE), (OR) and (wRM).

Proof. We show CM and $\sim$RM. The proof of the other postulates is similar (and follows from the results of [43, 49]).

For CM, assume that $\phi \mathrel{|\!\sim}^{\mathsf{K}}_{\preceq} \psi$ and $\phi \mathrel{|\!\sim}^{\mathsf{K}}_{\preceq} \gamma$. This means that (1) $\min_{\preceq}(\mathcal{V}(\phi)) \subseteq \mathcal{V}(\psi)$ and $\min_{\preceq}(\mathcal{V}(\phi)) \subseteq \mathcal{V}(\gamma)$. But then $\min_{\preceq}(\mathcal{V}(\phi \wedge \psi)) \subseteq \min_{\preceq}(\mathcal{V}(\phi)) \subseteq \mathcal{V}(\gamma)$ and thus $\phi \wedge \psi \mathrel{|\!\sim}^{\mathsf{K}}_{\preceq} \gamma$.

For $\sim$RM, suppose that $\phi \mathrel{|\!\sim}^{\mathsf{K}}_{\preceq} \gamma$ and $\phi \mathrel{|\!\not\sim}^{\mathsf{K}}_{\preceq} {\sim}\psi$. This means (1) $\min_{\preceq}(\mathcal{V}(\phi)) \subseteq \mathcal{V}(\gamma)$ and (2) there is some $v \in \min_{\preceq}(\mathcal{V}(\phi))$ s.t. $v(\psi) = \top$. Thus, $\min_{\preceq}(\bar{\mathcal{V}}(\phi \wedge \psi)) \subseteq \min_{\preceq}(\mathcal{V}(\phi))$. Since $\min_{\preceq_D}(\mathcal{V}(\phi)) \subseteq \mathcal{V}(\gamma)$, this implies $\min_{\preceq}(\mathcal{V}(\phi \wedge \psi)) \subseteq \mathcal{V}(\gamma)$, i.e. $\phi \wedge \psi \mathrel{|\!\sim}^{\mathsf{K}}_{\preceq} \gamma$. □

[18] Since $\preceq$ is a total order, we can equivalently replace any of the two existential quantifiers expressed by "for some" by a universal quantifier.

We show now that there are total preorders for which (RM) might be violated:

Example 21. Consider a preorder $\preceq$ over $\mathcal{V}(\{a,b\})$ s.t. $\mathsf{T}u \prec \mathsf{TT}, \mathsf{TF}$. Then:

- $a \mathrel{|\!\!\sim}^{\mathsf{K}}_{\preceq} \odot b$ (since $\min_{\preceq}(a \wedge \odot b) = \{\mathsf{T}u\}$, $\min_{\preceq}(a \wedge \sim \odot b) = \{\mathsf{TT}, \mathsf{TF}\}$ and $\mathsf{T}u \prec \mathsf{TT}$),
- $a \not\mathrel{|\!\!\sim}^{\mathsf{K}}_{\preceq} \neg b$ (since $\min_{\preceq}(a \wedge \neg b) = \{\mathsf{TF}\}$, $\min_{\preceq}(a \wedge \sim \neg b) = \{\mathsf{T}u\}$ and $\mathsf{TF} \not\prec \mathsf{T}u$), yet
- $a \wedge b \not\mathrel{|\!\!\sim}^{\mathsf{K}}_{\preceq} \odot b$ (since $\min_{\preceq}(a \wedge b \wedge \odot b) = \emptyset$).

Altogether, we can conclude that the basic ideas for obtaining nonmonotonic conditional inferences [43, 49] and defeasible inference relations known from propositional logic can be taken over to the three-valued setting, but some subtle differences (e. g. (wRM) vs (RM)) distinguish the resulting inference relations from their two-valued counterparts.

9.2 Defeasible conditional inference for ADFs

In this section, we study various ways of obtaining conditional inference relations on the basis of ADFs, and relate these conditional inference relations to revision and defeasible inference relations. A conditional inference relation should formalise expressions of the form "if ϕ then typically/normally ψ", where "typically" or "normally' aligns with the information expressed by an ADF.

We first define *static conditional inference relations*, which treat the interpretations selected by some semantics given an ADF as equally plausible, and any other interpretation as implausible or even impossible. An ADF D therefore implies a static conditional $\phi \Rightarrow \psi$ (given some semantics Sem), if there is an interpretation in $\mathsf{Sem}(D)$ that validates ϕ, and every interpretation in $\mathsf{Sem}(D)$ that validates ϕ also validates ψ.

Definition 21. Let an ADF $D = (\mathsf{At}, L, C)$, some semantics Sem and some $\phi, \psi \in \mathcal{L}^{\mathsf{K}}(\mathsf{At})$ be given. $D \mathrel{|\!\!\sim}^{\mathsf{st}}_{\mathsf{Sem}} \phi \Rightarrow \psi$ iff:

- there is some $v \in \mathsf{Sem}(D)$ s.t. $v(\phi) = \mathsf{T}$, and
- for every $v \in \mathsf{Sem}(D)$ s.t. $v(\phi) = \top$, $v(\psi) = \top$.

Example 22. Consider again D_1 from Example 3. We have e. g. $D_1 \mathrel{|\!\!\sim}^{\mathsf{st}}_{\mathsf{prf}} \top \Rightarrow c, \neg b \Rightarrow a, \neg a \Rightarrow b$.

Remark 5. In a two-valued propositional setting, static conditionals can be defined as follows (where $\delta \in \mathcal{L}$ functions as a background context): $\delta \mathrel{|\!\!\sim}^{\mathsf{st}} \phi \Rightarrow \psi$ iff $\delta \not\vdash \neg\phi$ (i.e. $\Omega(\delta) \cap \Omega(\phi) \neq \emptyset$) and $\delta \vdash \phi \rightarrow \psi$ (or equivalently: $\delta \wedge \phi \vdash \psi$). In other words, for two-valued propositional settings, static conditionals reduce to material conditionals.

Static conditional inference relations, however, are rather weak, since their antecedents are restricted to formulas that are implied by at least one interpretation selected by Sem. For example, $D_1 \not\mathrel{|\!\!\sim}^{\mathsf{st}}_{\mathsf{prf}} \neg c \Rightarrow \phi$ for any $\phi \in \mathcal{L}^{\mathsf{K}}(\{a,b,c\})$, not even $\neg c \Rightarrow \top$. Therefore, we introduce now dynamic conditional inference relations, based on revisions of ADFs. We construct a conditional inference relation for ADFs based on the *Ramsey test*, going back to [56]:

> If two people are arguing "If p, then q?" and are both in doubt as to p, they are adding p hypothetically to their stock of knowledge and arguing on that basis about q;

Based on this idea, we can simply state that the conditional $(\psi|\phi)$ is derivable from the ADF D, or, equivalently (in view of Fact 5) the conditional $(\psi|\phi)$ is valid in view of D, given a semantics Sem and some revision operator $\star$ iff ψ is derivable in the revised ADF $D \star \phi$ under the semantics Sem, resulting in the following definition of dynamic conditionals $D \mathrel{|\!\!\sim}^{\star}_{\mathsf{Sem}} (\psi|\phi)$:

Definition 22. Given an ADF D and a revision operator $\star$, $D \mathrel{|\!\!\sim}^{\star}_{\mathsf{Sem}} (\psi|\phi)$ is defined by $D \star \phi \mathrel{|\!\!\sim}^{\cap}_{\mathsf{Sem}} \psi$.

We can show that static conditional inference relations are weaker than dynamic conditional inference relations, according to any $\mathsf{ADF}^3_\star$-operator:

Proposition 17. Let an ADF D, some semantics Sem and an $\mathsf{ADF}^3_\star$-revision operator (for the semantics Sem) be given. Then $D \mathrel{|\!\!\sim}^{\mathsf{st}}_{\mathsf{Sem}} \phi \Rightarrow \psi$ implies $D \mathrel{|\!\!\sim}^{\star}_{\mathsf{Sem}} (\psi|\phi)$.

Proof. Suppose $\phi \mid\!\sim^{\mathsf{st}}_{\mathsf{Sem},D} \psi$, i. e. there is some $v \in \mathsf{Sem}(D)$ s.t. $v(\phi) = \top$, and for every $v \in \mathsf{Sem}(D)$ s.t. $v(\phi) = \top$, $v(\psi) = \top$. With $\mathsf{ADF}^3_\star 2$, $\mathsf{Sem}(D \star \phi) = \mathsf{Sem}(D) \cap \mathcal{V}(\psi)$. Since for every $v \in \mathsf{Sem}(D)$ s.t. $v(\phi) = \top$, $v(\psi) = \top$, this means $D \star \phi \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$ and thus $D \mid\!\sim^{\star}_{\mathsf{Sem}} (\psi|\phi)$. □

We first show that dynamic conditional inference relations based on revision of ADFs can be seen as a special case of three-valued conditional inference relations. We do this by showing that, given an $\mathsf{ADF}^3_\star$-operator $\star$, the corresponding total preorder $f^\star(D)$ gives rise to an inference relation $\mid\!\sim^{\mathsf{K}}_{f^\star(D)}$ equivalent to the conditionals $\mid\!\sim^{\star}_{\mathsf{Sem}}$-derivable from D.

Proposition 18. Given an ADF D, some semantics $\mathsf{Sem} \in \{\mathsf{prf}, \mathsf{grounded}\}$ and an $\mathsf{ADF}^3_\star$-operator $\star$ satisfying $(\mathsf{ADF}^3_\star 1)$-$(\mathsf{ADF}^3_\star 6)$, $D \mid\!\sim^{\star}_{\mathsf{Sem}} (\psi|\phi)$ iff $\phi \mid\!\sim^{\mathsf{K}}_{f^\star(D)} \psi$.

Proof. Let $\mathsf{Sem} \in \{\mathsf{prf}, \mathsf{grounded}\}$ With Theorem 4 and Theorem 5, $\phi \mid\!\sim^{\mathsf{Sem}}_{D,\star} \psi$ iff $\mathcal{V}(\psi) \supseteq \min_{f^\star(D)}(\mathcal{V}(\phi))$. With Fact 5, this implies $\phi \mid\!\sim^{\mathsf{Sem}}_{D,\star} \psi$ iff $\phi \mid\!\sim^{\mathsf{K}}_{f^\star(D)} \psi$. □

From this connection between dynamic conditionals and three-valued nonmonotonic inference relations, we can show that dynamic conditionals (or their equivalent formulation as nonmonotonic inference relations) satisfy all the KLM-postulates (as defined in Section 2.3) and (wRM):

Corollary 2. Let an ADF D, some semantics $\mathsf{Sem} \in \{\mathsf{prf}, \mathsf{grounded}\}$ and a $\mathsf{ADF}^3_\star$-operator $\star$ for Sem be given. Then $\mid\!\sim^{\mathsf{K}}_{f^\star(D)}$ satisfies (REF), (CUT), (CM), (RW), (LLE), (OR) and (wRM).

We illustrate these conditional inference relations with some conditionals derived from Example 14:

Example 23 (Example 14 continued)**.** Where $\star$ is the operator based on $\preceq^{\mathsf{prf,d+i}}_D$ and D_1 is as in Example 14, we see that e. g. $D_1 \mid\!\sim^{\mathsf{Sem}}_{\star} (\neg c|{\sim} c)$ and $D_1 \mid\!\sim^{\mathsf{Sem}}_{\star} (a|\neg a \wedge c)$. Notice that also e. g. $D_1 \mid\!\sim^{\mathsf{Sem}}_{\star} (C_c|c)$ (i. e. $D_1 \mid\!\sim^{\mathsf{Sem}}_{\star} (\neg a \vee \neg b|c)$) and $D_1 \mid\!\sim^{\mathsf{Sem}}_{\star} (c|C_c)$. In fact for any $s \in \{a, b, c\}$, $D_1 \mid\!\sim^{\mathsf{Sem}}_{\star} (C_s|s)$ and $D_1 \mid\!\sim^{\mathsf{Sem}}_{\star} (s|C_s)$.

We show in Example 24 that the syntactical structure of an ADF is not always respected by the resulting dynamic conditional inference relation:

Example 24. Let $D = (\{a\}, L, C_a = \neg a)$ and consider the preorder $u \prec \top \prec \bot$. It can be easily shown that there exists an i-modular mapping f s.t. $f(D) = \prec$. However, $a \not\mid\!\sim^{\mathsf{Sem}}_{D,\star} \neg a$ and $\neg a \not\mid\!\sim^{\mathsf{Sem}}_{D,\star} a$. A similar, but more involving example without a self-attacking argument for which a similar claim holds is: $D' = (\{a, b, c\}, L, C_a = \neg b, C_b = \neg c, C_c = \neg a)$.

9.3 Dynamical Conditionals Based on the Two-Valued Model Semantics

In this section, we look at the more basic case of dynamic conditionals based on two-valued model semantics.

We first notice that, given a $\mathsf{ADF}^2_\star$-operator $\star$ and an ADF D, where $\star$ is based on the total preorder $f^\star(D) = \preceq_D$, see Theorem 1, a dynamical conditional consequence relation $D \mid\!\sim^{\mathsf{Sem}}_{\star}$ can be equivalently represented as conditional inference relation induced by the total preorder $\preceq_D$ over Ω. Given a $\mathsf{ADF}^2_\star$-operator satisfying $(\mathsf{ADF}^2_\star 1)$-$(\mathsf{ADF}^2_\star 6)$, we denote by $f^\star(D)$ the total preorder over Ω induced by $\star$ and D as in Theorem 1.[19]

Proposition 19. Given a semantics $\mathsf{Sem} \in \{\mathsf{2val}, \mathsf{stable}\}$, an ADF D and a $\mathsf{ADF}^2_\star$-operator $\star$ satisfying $(\mathsf{ADF}^2_\star 1)$-$(\mathsf{ADF}^2_\star 6)$, $D \mid\!\sim^{\mathsf{Sem}}_{\star} (\psi|\phi)$ iff $\phi \mid\!\sim_{f^\star(D)} \psi$.

Proof. Suppose $\star$ is an $\mathsf{ADF}^2_\star$-operator satisfying $(\mathsf{ADF}^2_\star 1)$-$(\mathsf{ADF}^2_\star 6)$. By Theorem 1 there is a total preorder $\preceq_\star$ s.t. $\mathsf{2mod}(D \star \psi) = \min_{f(D)}(\mathsf{Mod}(\psi))$ for any $\psi \in \mathcal{L}$. [$\Rightarrow$]: suppose $\phi \mid\!\sim^{\mathsf{Sem}}_{D,\star} \psi$, i. e. $\mathsf{Mod}(\psi) \supseteq \min_{\preceq}(\mathsf{Mod}(\phi))$. This means that $\min_{f(D)}(\mathsf{Mod}(\phi \wedge \psi)) \preceq \min_{f(D)}(\mathsf{Mod}(\phi \wedge \neg\psi))$ and thus $\phi \mid\!\sim_{f^\star(D)} \psi$.
[$\Leftarrow$]: analogous. □

We notice the following property:

[19] Notice that the semantics Sem relative to which a $\mathsf{ADF}^2_\star$-operator is defined are implicitly taken into account in $f^\star(D)$, in the sense that the realisability of this semantics will be taken into account in the additional conditions on the total preorder.

Corollary 3. Let an ADF D and an $\mathsf{ADF}^2_\star$-operator $\star$ that satisfies ($\mathsf{ADF}^2_\star 1$)-($\mathsf{ADF}^2_\star 6$) be given. Then $\mid\!\sim^{\mathsf{Sem}}_{D,\star}$ satisfies (REF), (CUT), (CM), (RW), (LLE), (OR) and (RM).

Proof. From Proposition 19 it follows that $\mid\!\sim^{\mathsf{Sem}}_{D,\star} = \mid\!\sim_{f^\star(D)}$. With Proposition 1 we can derive that $\mid\!\sim^{\mathsf{Sem}}_{D,\star}$ satisfies (REF), (CUT), (CM), (RW), (LLE), (OR) and (RM). □

Thus, in contradistinction to the three-valued case, conditional inference based on ADFs w.r.t. two-valued semantics is a special case of preferential inference.

10 Related Works

The contributions of this paper are summarised as follows: (1) the definition of revision operators for ADFs under all major semantics, and their semantic representation in terms of total preorders over interpretations, and (2) the definition of defeasible conditional inference for ADFs on the basis of revision operators. We discuss works related to our contributions, in the sense that they either treat revision of argumentative formalisms or conditional inference for argumentative formalisms.

10.1 Revision of ADFs and Argumentation Frameworks

Revision of ADFs is investigated in the work by Linsbichler and Woltran [47]. In that paper, revision of ADFs by other ADFs are defined. Conceptually, our approach is able to capture the approach by [47] since we allow for revisions of ADFs under three-valued semantics by any $\mathcal{L}^{\mathsf{K}}$, which allows to express revision by a set of interpretations $\mathcal{V}'$ as revision by the formula $\bigvee \mathcal{V}'$. Technically, there is some incomparability between our approach and that of [47] caused by the difference in the type of revision we consider. In particular, there are differences in the way the issue of realisability is handled. We have chosen to handle this issue by ensuring that any subset of a $\preceq_D$-layer is realisable under a given semantics, whereas [47] handles this issue by defining revisions of the ADF D by another ADF as $f_{\mathsf{Sem}}(\min_{\preceq_D} \mathsf{Sem}(F))$, where the function $f_{\mathsf{Sem}}(\mathcal{V}')$ returns $\mathcal{V}'$ if it is realisable under $\mathcal{V}'$ and the interpretation v_u[20] otherwise.

Revisions of abstract argumentation frameworks are considered in many works, including [9, 25, 11, 3, 46, 19, 18, 48, 45]. The approach closest to revisions as defined in our paper is that of [20], where revisions of argumentation frameworks are also defined indirectly by specifying the set of extensions (according to some semantics). In more detail, in [20], revisions of argumentation frameworks by both propositional formulas and other argumentation frameworks (represented as sets of extensions according to some semantics) are defined and characterised. Thus, conceptually, we provide generalisations of both these kinds of revisions, as we allow for revisions by any formula in the language $\mathcal{L}^{\mathsf{K}}$, which allows to represent sets of extensions. With regards to the differences between revision of abstract argumentation frameworks and abstract dialectical frameworks, we conjecture that we can characterise revisions of abstract argumentation frameworks as revisions of ADFs. Indeed, since ADFs are strictly more expressive than abstract argumentation frameworks [61], our work subsumes, conceptually, the work of [20].

10.2 Conditional Inference in Argumentation

A number of works have studied the conditional inferential behaviour of formal argumentation formalisms. In structured argumentation, there are a number of works that study KLM-like properties of argumentative inference relations [7, 34, 38, 12, 13, 44]. These work differ both in host formalism (various formalisms for structured argumentation versus ADFs) and the way conditional inference is defined. Whereas we define conditional inference both by static conditionals and by using the Ramsey-test, these works consider a conditional $(\psi|\phi)$ to be justified if, after the addition of ϕ to the knowledge base (sometimes as a strict premise, sometimes as a defeasible premise), ψ is derivable according to the chosen argumentative inference relation.

In [57, Chapter 3] a type of inference for abstract argumentation frameworks is defined which is not unrelated to the conditional inference relations studied in this paper. There, inference relations are based on *interventions* of argumentation frameworks, inspired by interventions in Bayesian networks. Interventions of argumentation frameworks allow to enforce a labelling status of an argument by adding new arguments that attack the argument

[20] Recall that $v_u(s) = u$ for every $s \in \mathsf{At}$.

which labelling status is to be intervened. Given an argumentation framework F, an inference relation based on such interventions is then defined by stating that $\Psi \mid\!\sim^{F}_{\sigma} \phi$ iff after the intervention Ψ, ϕ is true according to all σ-labellings of the argumentation framework that is the result of the intervention Ψ on F. [57] studies several properties of such inference relations, include the KLM-properties, for which it is shown that for restricted classes of interventions, some semantics satisfy cautious monotony, cut and rational monotony. Furthermore, for subclasses of argumentation frameworks, such as odd-cycle-free, even-cycle-free and acyclic argumentation frameworks, stronger properties are shown. Another approach is that of *conditional acceptance functions* [5], where, given an argumentation framework, the usual labelling semantics are changed as to account for abductive or counterfactual reasoning. In more detail, conditional acceptance functions take, in addition to an argumentation framework, as an input a set of labels assigned to arguments. Such input labels can serve as explanation for a possibly different argument's status. For example, if an argument has as an input label `out`, an argument attacked by this label need not be defended from this argument.

In [6] conditional inference relations for abstract argumentation frameworks are defined on the basis of a propositional language built up from atoms $\mathtt{in}_s$, $\mathtt{out}_s$ and $\mathtt{u}_s$ for every argument s, which encode argument labels. Nonmonotonic inference relations are then defined semantically by a total preorder over models for this language by preferring models that model labellings that "satisfy better" the constraints of a selected semantics (given the argumentation framework under consideration). For example, under the admissible semantics, models are compared w.r.t. the degree to which they defend the arguments labelled `in` by them. Based on the resulting preorder, the conditional $(\psi|\phi)$ then holds if ψ holds in all minimal models of ϕ. We notice a couple of differences with our work: (1) we allow for any total preorder, (2) we work with the more general ADFs, (3) we study the KLM-properties, in contradistinction to [6], (4) we use a language where the arguments form atoms, whereas [6] use labels for arguments, (5) the selection of ϕ-models in which inference of ψ is checked to ascertain the acceptability of $(\psi|\phi)$ might not be realisable by a revised argumentation framework in [6], whereas in our framework this is guaranteed.

10.3 Translations from ADFs into Conditional Logics

In [36, 37], translations of ADFs in conditional logics have been investigated. Here, we show that these translations can also be used to define revision operators.

We recall the following translations [36, 37] from ADFs in conditional knowledge bases (where $D = (\mathsf{At}, L, C)$:

- $\Theta_1(D) = \{(s|C_s) \mid s \in \mathsf{At}\}$
- $\Theta_2(D) = \{(C_s|s) \mid s \in \mathsf{At}\}$
- $\Theta_3(D) = \Theta_1(D) \cup \Theta_2(D)$
- $\Theta_4(D) = \Theta_1(D) \cup \{(\neg s|\neg C_s) \mid s \in \mathsf{At}\}$
- $\Theta_5(D) = \{((C_s \equiv s)|\top) \mid s \in \mathsf{At}\}$
- $\Theta_6(D) = \Theta_2(D) \cup \{(\neg C_s|\neg s) \mid s \in \mathsf{At}\}$.
- $\Theta_7(D) = \{(\neg s|\neg C_s) \mid s \in \mathsf{At}\} \cup \{(\neg C_s|\neg s) \mid s \in \mathsf{At}\}$.

We will make use of the following propositions [37]:

Proposition 20 ([37])**.** $\Theta_5(D) = (\Theta_5(D))_0$ for any ADF D for which $\mathsf{2mod}(D) \neq \emptyset$.

Definition 23 ([37])**.** An ADF $D = (\mathsf{At}, L, C)$ is:

- *non-refuting* if there is no $s \in \mathsf{At}$ s.t. $\sqcap_i \mathsf{2mod}(s) = \bot$.
- *non-validating* if there is no $s \in \mathsf{At}$ s.t. $\sqcap_i \mathsf{2mod}(s) = \top$.

Proposition 21 ([37])**.** For any ADF D:

- If D is consistent $\Theta_3(D) = (\Theta_3(D))_0$.
- If D is non-refuting and non-validating, $\Theta_i(D) = (\Theta_i(D))_0$ for $i \in \{4, 6\}$.

- If D is non-validating, $\Theta_7(D) = (\Theta_7(D))_0$.

Proposition 22 ([37])**.** For any $i \in \{3, 4, 5, 6, 7\}$, if $\Theta_i(D)$ is consistent then $\omega \in (\kappa^Z_{\Theta(D)})^{-1}(0) = \mathsf{2mod}(D)$.

Example 25. We illustrate revisions of ADFs on the basis of translations from ADFs into conditional logic by looking back at D_1 from Example 3. We first consider the preorder induced by $\kappa^Z_{\Theta_5(D)}$:

ω	$\kappa^Z_{\Theta_5(D)}$	ω	$\kappa^Z_{\Theta_5(D)}$	ω	$\kappa^Z_{\Theta_5(D)}$	ω	$\kappa^Z_{\Theta_5(D)}$
abc	1	$ab\overline{c}$	1	$a\overline{b}c$	0	$a\overline{b}\overline{c}$	1
$\overline{a}bc$	0	$\overline{a}b\overline{c}$	1	$\overline{a}\overline{b}c$	1	$\overline{a}\overline{b}\overline{c}$	1

Notice that a revision with $\neg c$ will give rise to an ADF with the following two-valued models: $\{ab\overline{c}, a\overline{b}\overline{c}, \overline{a}b\overline{c}, \overline{a}\overline{b}\overline{c}\}$. Using the method above, the following ADF will have such two-valued models: $D \star \neg c = (\{a, b, c\}, L, C')$ with:

$$\begin{array}{rcl} C'_a & = & a\overline{b}\overline{c} \vee ab\overline{c} \vee \overline{a}bc \vee \overline{a}\overline{b}c \\ C'_b & = & \overline{a}b\overline{c} \vee ab\overline{c} \vee a\overline{b}c \vee \overline{a}\overline{b}c \\ C'_c & = & \perp \end{array}$$

These conditions can be simplified as follows:

$$\begin{array}{rcl} C'_a & = & a\overline{c} \vee \overline{a}c \\ C'_b & = & b\overline{c} \vee \overline{b}c \\ C'_c & = & \perp \end{array}$$

We now show that the revisions associated with the assignment of preorders induced by $\kappa^Z_{\Theta_i(D)}$ are well-behaved. It turns out that they are. We first define $f_{\Theta_i}(D)$ as the function assigning $\preceq_{\Theta_i(D)}$ where $\omega \preceq_{\Theta_i(D)} \omega'$ iff $\kappa^Z_{\Theta_i(D)}(\omega) \leq \kappa^Z_{\Theta_i(D)}(\omega')$.

Proposition 23. Let some $3 \leq i \leq 6$ be given. $f_{\Theta_i}(D)$ is faithful w.r.t. the two-valued model semantics for the class of ADFs for which $\Theta_i(D)$ is consistent.

Proof. Since $\kappa^Z_{\Theta_i(D)}(\omega) = 0$ iff $\omega \in \mathsf{2mod}(D)$ by Propositions 22, 22 and 22, it immediately follows that the $\kappa^Z_{\Theta_i(D)}(\omega)$-maximal elements coincide with $\mathsf{2mod}(D)$, which suffices to show items 1, 2 and 3 of Definition 7. □

We can show that for any assignment of preorders faithful w.r.t. the two-valued model semantics, $\preceq_D$ is a refinement of the preorder $\kappa^Z_{\Theta_5(D)}$ induced by the translation Θ_5, in the sense that if $\kappa^Z_{\Theta_5(D)}(\omega) < \kappa^Z_{\Theta_5(D)}(\omega')$ then $\omega \prec \omega'$.

Proposition 24. Let a semantics Sem and an assignment f of preorders $\preceq_D$ faithful w.r.t. the two-valued model semantics be given. Then $\kappa^Z_{\Theta_5(D)}(\omega) < \kappa^Z_{\Theta_5(D)}(\omega')$ implies $\omega \prec_D \omega'$.

Proof. Consider some $\omega, \omega' \in \Omega(\mathsf{At})$ and suppose $\kappa^Z_{\Theta_5(D)}(\omega) < \kappa^Z_{\Theta_5(D)}(\omega')$. By Proposition 20, this means that $\kappa^Z_{\Theta_5(D)}(\omega) = 0$ and $\kappa^Z_{\Theta_5(D)}(\omega') = 1$. By Propositions 22, 22 and 22, this means that $\omega \in \mathsf{2mod}(D)$ and $\omega' \notin \mathsf{2mod}(D)$. Since $\preceq$ is faithful w.r.t. D and $\mathsf{2mod}$, $\omega \prec \omega'$. □

We can also show that for any ADF D, $\mid\!\sim_{\kappa^Z_{\Theta_5(D)}}$ is the most skeptical conditional inference relation for D based on the two-valued model semantics:

Proposition 25. Let an ADF D and a revision operator $\star$ be given. If $\phi \mid\!\sim_{\kappa^Z_{\Theta_5(D)}} \psi$ then $\phi \mid\!\sim^{\mathsf{2mod}}_{D,\star} \psi$.

Proof. Suppose $\phi \mid\!\sim_{\kappa^Z_{\Theta_5(D)}} \psi$, i.e. $\kappa^Z_{\Theta_5(D)}(\phi \wedge \psi) < \kappa^Z_{\Theta_5(D)}(\phi \wedge \neg\psi)$. By Proposition 20, this means that there is some $\omega \in \Omega(\mathsf{At})$ s.t. $\kappa^Z_{\Theta_5(D)}(\omega) = 0$ and $\omega \models \phi \wedge \psi$ and for every $\omega' \in \Omega(\mathsf{At})$ s.t. $\omega' \models \phi \wedge \neg\psi$, $\kappa^Z_{\Theta_5(D)}(\omega') = 1$. By Proposition 24, for any $\preceq$ that is faithful w.r.t. D and $\mathsf{2mod}$, this means that there is an $\omega \in \Omega(\mathsf{At})$ s.t. $\omega \models \phi \wedge \psi$ and $\omega \prec \omega'$ for any $\omega' \in \Omega(\mathsf{At})$ s.t. $\omega' \models \phi \wedge \neg\psi$. This implies that $\min_{\preceq}(\omega \in \Omega(\mathsf{At}) \mid \omega \models \phi) \models \psi$. Thus (by definition of a revision operator $\star$ for ADFs), for any revision $\star$, $D \star \phi \mid\!\sim^{\mathsf{2mod}}_{D,\star} \psi$. □

The above proposition shows that the fact that our translations induce only a limited conditional structure (since e.g. $\Theta_5 = (\Theta_5)_0$, cf. Proposition 20, and similarly for other translations, cf. Theorem 21) can also be viewed positively. Indeed, this lack of conditional depth induced by the translations is exactly what allows these translations to be seen as a *core logic* for dynamical conditional reasoning based on ADFs.

11 Conclusion

In this paper, we defined dynamic conditional inference relations for ADFs based on the Ramsey test, and developed a new approach to revision of an ADF by formulas to achieve this. We have shown that such conditional inference relations satisfy all the usual rationality postulates for conditional inferences and extend static conditionals but also give rise to subtle differences with the propositional case, as witnessed e. g. by the (wRM)-postulate and the negative results on possibilistic revision operators. What comes out clearly from this work is that revision, or more generally belief change, is the platform that allowed us to bridge the gap between argumentative reasoning and conditional inference. As such, we hope that this work will serve as an inspiration for further investigations into the combination and cross-fertilization between argumentative and nonmonotonic conditional reasoning. Indeed, rather than a definitive statement on dynamic conditional inference for ADFs, we see this paper as an anchor point for further research on revision, nonmonotonic inference and dynamic conditional argumentative reasoning. When generalising these interconnected concepts, there are many choices to be made, such as which "monotonic base logic" to use (in our case: K), which postulates for revision to use (e. g. the canonical approach of [40] vs alternative types of revision [26]), how exactly to adapt the postulates for revision and the corresponding faithful mappings (e. g. use equality of extensions vs strong equivalence). Even though we believe that the choices we made are well-motivated, they are clearly not the only viable ones. We have shown how some of these choices can be easily adapted (e. g. going from equivalence to strong equivalence) whereas other choices are less alterable (going from Kleene's logic K to possibilistic logic).

We summarise our findings on revisions in the following table:

Type of Revision	Semantics	Postulates	Requirement on Mapping	Main Result
$\mathcal{L}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathcal{L}(\mathsf{At})$	n.a.	$(\mathsf{R1})$-$(\mathsf{R6})$	faithful	[40]
$\mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathcal{L}(\mathsf{At})$	$\mathsf{2mod}$	$(\mathsf{ADF}^2_\star 1)$-$(\mathsf{ADF}^2_\star 6)$	faithful w.r.t. $\mathsf{2mod}$	Cor. 1
$\mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathcal{L}(\mathsf{At})$	stable	$(\mathsf{ADF}^2_\star 1)$-$(\mathsf{ADF}^2_\star 6)$	$\top$-modular faithful w.r.t. stable	Cor. 3
$\mathcal{L}^{\mathsf{K}}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathcal{L}^{\mathsf{K}}(\mathsf{At})$	n.a.	$(\mathsf{R}^3 1)$-$(\mathsf{R}^3 1)$	faithful (over $\mathcal{V}(\mathsf{At})$)	Cor. 7
$\mathfrak{D}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathcal{L}^{\mathsf{K}}(\mathsf{At})$	prf	$(\mathsf{ADF}^3_\star 1)$-$(\mathsf{ADF}^3_\star 6)$	i-modular faithful w.r.t. prf	Cor. 4
$\mathfrak{D}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathcal{L}^{\mathsf{K}}(\mathsf{At})$	$\mathsf{grounded}$	$(\mathsf{ADF}^3_\star 1)$-$(\mathsf{ADF}^3_\star 6)$	anti-symmetric faithful w.r.t. $\mathsf{grounded}$	Cor. 4

In future work, we plan to look deeper into the semantical nature of revisions defined in this work. Indeed, a revised ADF is only defined in terms of its models (according to a chosen semantics). What is not specified is how we can obtain the revised ADF in terms of changes (be it revisions or otherwise) of the original ADF, and in particular its conditions. We have shown that this cannot be done straightforwardly in terms of revision of the acceptance conditions. However, there might be more sophisticated workarounds for this.

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Appendix: Propositional Revision under Three-valued Semantics

In this section we show that three-valued revisions of formulas in the language $\mathcal{L}^{\mathsf{K}}$ can be characterized in terms of a selection of minimal worlds according to a total preoder $\preceq$ over the three-valued interpretations, completely analogous to the two-valued case (see Section 2.4, in particular Theorem 1).

Definition 24. An operator $\star : \mathcal{L}^{\mathsf{K}} \times \mathcal{L}^{\mathsf{K}} \to \mathcal{L}^{\mathsf{K}}$ is a *trivalent revision operator* iff $\star$ satisfies (for any $\phi, \psi, \mu \in \mathcal{L}^{\mathsf{K}}$):

($\mathsf{R}^3 1$) $\phi \star \psi \vdash_{\mathsf{K}} \psi$

($\mathsf{R}^3 2$) If $\phi \wedge \psi \not\vdash_{\mathsf{K}} \bot$ then $\phi \star \psi \equiv_{\mathsf{K}} \phi \wedge \psi$

($\mathsf{R}^3 3$) If $\psi \not\vdash_{\mathsf{K}} \bot$ then $\phi \star \psi \not\vdash_{\mathsf{K}} \bot$

($\mathsf{R}^3 4$) If $\phi \equiv_{\mathsf{K}} \phi'$ and $\psi \equiv_{\mathsf{K}} \psi'$ then $\phi \star \psi \equiv_{\mathsf{K}} \phi' \star \psi'$.

($\mathsf{R}^3 5$) $(\phi \star \psi) \wedge \mu \vdash_{\mathsf{K}} \phi \star (\psi \wedge \mu)$.

($\mathsf{R}^3 6$) If $(\phi \star \psi) \wedge \mu \not\vdash_{\mathsf{K}} \bot$ then $\phi \star (\psi \wedge \mu) \vdash_{\mathsf{K}} (\phi \star \psi) \wedge \mu$

Remark 6. These properties can be characterized equivalently in a more semantical way as follows:

($\mathsf{R}^3_{\mathsf{s}} 1$) $\mathcal{V}(\phi \star \psi) \subseteq \mathcal{V}(\psi)$.

($\mathsf{R}^3_{\mathsf{s}} 2$) If $\mathcal{V}(\phi) \cap \mathcal{V}(\psi) \neq \emptyset$ then $\mathcal{V}(\phi \star \psi) = \mathcal{V}(\phi) \cap \mathcal{V}(\psi)$.

($\mathsf{R}^3_{\mathsf{s}} 3$) If $\mathcal{V}(\psi) \neq \emptyset$ then $\mathcal{V}(\phi \star \psi) \neq \emptyset$.

($\mathsf{R}^3 4$) If $\mathcal{V}(\phi) = \mathcal{V}(\phi')$ and $\mathcal{V}(\psi) = \mathcal{V}(\psi')$ then $\mathcal{V}(\phi \star \psi) = \mathcal{V}(\phi' \star \psi')$.

($\mathsf{R}^3_{\mathsf{s}} 5$) $\mathcal{V}(\phi \star \psi) \cap \mathcal{V}(\mu) \subseteq \mathcal{V}(\phi \star (\psi \wedge \mu))$.

($\mathsf{R}^3_{\mathsf{s}} 6$) If $\mathcal{V}(\phi \star \psi) \cap \mathcal{V}(\mu) \neq \emptyset$, then $\mathcal{V}(\phi \star (\psi \wedge \mu)) \subseteq \mathcal{V}(\phi \star \psi) \cap \mathcal{V}(\mu)$.

Definition 25. Given a set of atoms At, a mapping $f : \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$ associating a total preorder $\preceq_\phi$ to every formula $\phi \in \mathcal{L}^{\mathsf{K}}(\mathsf{At})$ and every $v_1, v_2 \in \mathcal{V}(\mathsf{At})$: is a *faithful mapping* if, for every $\phi \in \mathcal{L}^{\mathsf{K}}$:

1. if $v_1 \in \mathcal{V}(\phi)$ then $v_1 \preceq_\phi v_2$; and
2. if $v_1 \in \mathcal{V}(\phi)$ and $v_2 \notin \mathcal{V}(\phi)$ then $v_1 \prec_\phi v_2$; and
3. if $\phi \equiv_{\mathsf{K}} \phi'$ then $\preceq_\phi = \preceq_{\phi'}$.

To show Theorem 7, we show the following two propositions:

Proposition 26. Let At be a finite set of atoms and $f : \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$ a faithful mapping. If $\star : \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathcal{L}^{\mathsf{K}}(\mathsf{At})$ is defined by $\mathcal{V}(\phi \star \psi) = \min_{\preceq_\phi}(\mathcal{V}(\psi))$, then $\star$ satisfies ($\mathsf{R}^3 1$)-($\mathsf{R}^3 6$).

Proof. Assume $f : \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \wp(\mathcal{V}(\mathsf{At}) \times \mathcal{V}(\mathsf{At}))$ is a faithful mapping and let $\star : \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathcal{L}^{\mathsf{K}}(\mathsf{At})$ be defined by $\mathcal{V}(\phi \star \psi) = \min_{\preceq_\phi}(\mathcal{V}(\psi))$. We show that $\star$ satisfies ($\mathsf{R}^3_{\mathsf{s}} 1$)-($\mathsf{R}^3_{\mathsf{s}} 6$) (according to the formulation in Remark 6).

- ($\mathsf{R}^3_{\mathsf{s}} 1$): Clear, since by definition, $\mathcal{V}(\phi \star \psi) = \min_{\preceq_D}(\mathcal{V}(\psi)) \subseteq \mathcal{V}(\psi)$.
- ($\mathsf{R}^3_{\mathsf{s}} 2$): Suppose $\mathcal{V}(\phi) \cap \mathcal{V}(\psi) \neq \emptyset$. Since $\preceq_\phi$ is faithful, $\mathcal{V}(\phi) \cap \mathcal{V}(\psi) = \min_{\preceq_\phi}(\mathcal{V}(\psi))$.
- ($\mathsf{R}^3_{\mathsf{s}} 3$): Suppose $\mathcal{V}(\psi) \neq \emptyset$. As $\preceq_\phi$ is transitive and At is finite, $\min_{\preceq_\phi}(\mathcal{V}(\psi)) \neq \emptyset$.
- ($\mathsf{R}^3_{\mathsf{s}} 4$): Suppose $\phi \equiv_{\mathsf{K}} \phi'$ and $\psi \equiv_{\mathsf{K}} \psi'$. Since f is faithful, $\preceq_\phi = \preceq_{\phi'}$. Since $\psi \equiv_{\mathsf{K}} \psi'$ implies $\mathcal{V}(\psi) = \mathcal{V}(\psi')$, we see that
$$\mathcal{V}(\phi \star \psi) = \min_{\preceq_\phi}(\mathcal{V}(\psi)) = \min_{\preceq_{\phi'}}(\mathcal{V}(\psi')) = \mathcal{V}(\phi' \star \psi')$$
and thus $\phi \star \psi = \phi' \star \psi'$.

- ($\mathsf{R}^3_\mathsf{s}5$) and ($\mathsf{R}^3_\mathsf{s}6$): The case where $\mathcal{V}(\phi\star\psi)\cap\mathcal{V}(\mu)=\emptyset$ is trivial. Suppose therefore that $\mathcal{V}(\phi\star\psi)\cap\mathcal{V}(\mu)\neq\emptyset$ and et $v\in\mathcal{V}(\phi\star\psi)\cap\mathcal{V}(\mu)$. Suppose furthermore towards a contradiction that $v\notin\mathcal{V}(\phi\star(\psi\wedge\mu))=\min_{\preceq_\phi}\mathcal{V}(\psi\wedge\mu)$, i.e. there is some $v'\in\mathcal{V}(\psi\wedge\mu)$ s.t. $v'\preceq_\phi v$. Since $\mathcal{V}(\psi\wedge\mu)\subseteq\mathcal{V}(\psi)$, this contradicts $v\in\mathcal{V}(\phi\star\psi)\cap\mathcal{V}(\mu)=\min_{\preceq_\phi}(\mathcal{V}(\psi)\cap\mathcal{V}(\mu))$. Thus, we have shown that $\mathcal{V}(\phi\star\psi)\cap\mathcal{V}(\mu)\subseteq\mathcal{V}(\phi\star(\psi\wedge\mu))$.

  Suppose now that $v\in\mathcal{V}(\phi\star(\psi\wedge\mu))=\min_{\preceq_\phi}(\mathcal{V}(\psi\wedge\mu))$. Notice that $v\in\mathcal{V}(\psi)\cap\mathcal{V}(\mu)$. Since we assumed $\mathcal{V}(\phi\star\psi)\cap\mathcal{V}(\mu)\neq\emptyset$, there is a $v'\in\mathcal{V}(\phi\star\psi)\cap\mathcal{V}(\mu)=\min_{\preceq_\phi}(\mathcal{V}(\psi))\cap\mathcal{V}(\mu)$. Since $v\in\min_{\preceq_\phi}(\mathcal{V}(\psi\wedge\mu))$, we assumed $\preceq_\phi$ to be total and $\min_{\preceq_\phi}(\mathcal{V}(\psi))\cap\mathcal{V}(\mu)\subseteq\mathcal{V}(\psi)\cap\mathcal{V}(\mu)$, $v\preceq_\phi v'$. Thus, $v\in\min_{\preceq_\phi}(\mathcal{V}(\psi))$. ☐

Proposition 27. Let a set of atoms At, and a trivalent revision operator $\star$ be given. Then there is an faithful mapping $f:\mathcal{L}^{\mathsf{K}}(\mathsf{At})\to\wp(\mathcal{V}(\mathsf{At})\times\mathcal{V}(\mathsf{At}))$ s.t. for every $\phi\in\mathcal{L}^{\mathsf{K}}(\mathsf{At})$, $\mathcal{V}(\phi\star\psi)=\min_{\preceq_\phi}(\mathcal{V}(\psi))$.

Proof. Assume $\phi\in\mathcal{L}^{\mathsf{K}}(\mathsf{At})$. We define $\preceq_\phi$ as follows: $v_1\preceq_\phi v_2$ iff $v_1\in\mathcal{V}(\phi\star\mathsf{form}(v_1)\vee\mathsf{form}(v_2))$. We first show that $\preceq_\phi$ is a total preorder:

Totality: consider some $v_1,v_2\in\mathcal{V}(\mathsf{At})$. Clearly, $\mathcal{V}(\mathsf{form}(v_1)\vee\mathsf{form}(v_2))=\{v_1,v_2\}$. By ($\mathsf{R}^3_\mathsf{s}1$), $\mathcal{V}(\phi\star\mathsf{form}(v_1)\vee\mathsf{form}(v_2))\subseteq\mathcal{V}(\mathsf{form}(v_1)\vee\mathsf{form}(v_2))=\{v_1,v_2\}$. By ($\mathsf{R}^3_\mathsf{s}3$) and since $\mathcal{V}(\mathsf{form}(v_1)\vee\mathsf{form}(v_2))\neq\emptyset$, $\mathcal{V}(\phi\star\mathsf{form}(v_1)\vee\mathsf{form}(v_2))\neq\emptyset$ and thus $v_1\in\mathcal{V}(\phi\star\mathsf{form}(v_1)\vee\mathsf{form}(v_2))$ or $v_2\in\mathcal{V}(\phi\star\mathsf{form}(v_1)\vee\mathsf{form}(v_2))$ which implies $v_1\preceq_\phi v_2$ or $v_2\preceq_\phi v_1$.

Transitivity: Suppose $v_1\preceq_\phi v_2$ and $v_2\preceq_\phi v_3$. We show that $v_1\preceq_\phi v_3$. By ($\mathsf{R}^3_\mathsf{s}1$) and ($\mathsf{R}^3_\mathsf{s}3$) we know that $\emptyset\neq\mathcal{V}(\phi\star\bigvee_{i=1}^3\mathsf{form}(v_i))\subseteq\{v_1,v_2,v_3\}$.

1. Suppose first that $\mathcal{V}(\phi\star\bigvee_{i=1}^3\mathsf{form}(v_i))\cap\{v_1,v_2\}=\emptyset$. This means that $\mathcal{V}(\phi\star\bigvee_{i=1}^3\mathsf{form}(v_i))=\{v_3\}$. By ($\mathsf{R}^3_\mathsf{s}5$) and ($\mathsf{R}^3_\mathsf{s}6$), $\mathcal{V}(\phi\star\Big((\bigvee_{i=1}^3\mathsf{form}(v_i)\wedge(\mathsf{form}(v_2)\vee\mathsf{form}(v_3))\Big))=\mathcal{V}(\phi\star\bigvee_{i=1}^3\mathsf{form}(v_i))\cap\{v_2,v_3\}$. Thus, $\mathcal{V}(\phi\star\mathsf{form}(v_2)\vee\mathsf{form}(v_3))=\{v_3\}$ and thus $v_3\prec_\phi v_2$, which contradicts $v_2\preceq_\phi v_3$.
2. Suppose now that $\mathcal{V}(\phi\star\bigvee_{i=1}^3\mathsf{form}(v_i))\cap\{v_1,v_2\}\neq\emptyset$. Since $v_1\preceq_\phi v_2$, $v_1\in\mathcal{V}(\phi\star\mathsf{form}(v_1)\vee\mathsf{form}(v_2))$. Using ($\mathsf{R}^3_\mathsf{s}5$) and ($\mathsf{R}^3_\mathsf{s}6$) in a similar way as for the previous case, we can show that $v_1\in\mathcal{V}(\phi\star\mathsf{form}(v_1)\vee\mathsf{form}(v_3))$ which implies $v_1\preceq_\phi v_3$.

Reflexivity: By ($\mathsf{R}^3_\mathsf{s}1$), $\mathcal{V}(\phi\star\mathsf{form}(v_1)\vee\mathsf{form}(v_1))\subseteq\{v_1\}$ and thus (since with ($\mathsf{R}^3_\mathsf{s}3$) $\mathcal{V}(\phi\star\mathsf{form}(v_1)\vee\mathsf{form}(v_1))\neq\emptyset$), $\mathcal{V}(\phi\star\mathsf{form}(v_1)\vee\mathsf{form}(v_1))=\{v_1\}$ and thus $v_1\preceq_\phi v_1$.

Now we show that $\preceq_\phi$ is faithful:

1. Suppose $v_1,v_2\in\mathcal{V}(\phi)$. By ($\mathsf{R}^3_\mathsf{s}2$), $\mathcal{V}(\phi\star\mathsf{form}(v_1)\vee\mathsf{form}(v_2))=\mathcal{V}(\phi)\cap\{v_1,v_2\}=\{v_1,v_2\}$. Thus, by definition of f, $v_1\preceq_\phi v_2$ and $v_2\preceq_\phi v_1$.
2. Suppose $v_1\in\mathcal{V}(\phi)$ and $v_2\notin\mathcal{V}(\phi)$. By ($\mathsf{R}^3_\mathsf{s}2$), $\mathcal{V}(\phi\star\mathsf{form}(v_1)\vee\mathsf{form}(v_2))=\mathcal{V}(\phi)\cap\{v_1,v_2\}=\{v_1\}$. Thus, by definition of f, $v_1\prec_\phi v_2$.
3. Suppose $\phi\equiv_{\mathsf{K}}\phi'$. By ($\mathsf{R}^3_\mathsf{s}4$), $\phi\star(\mathsf{form}(v_1)\vee\mathsf{form}(v_2))=\phi'\star(\mathsf{form}(v_1)\vee\mathsf{form}(v_2))$ for any $v_1,v_2\in\mathcal{V}(\mathsf{At})$ and thus by definition of f, $\preceq_\phi=\preceq_{\phi'}$. ☐

Theorem 7. $\star:\mathcal{L}^{\mathsf{K}}\times\mathcal{L}^{\mathsf{K}}\to\mathcal{L}^{\mathsf{K}}$ is a trivalent revision operator $\star$ iff there exists a function $f:\mathcal{L}^{\mathsf{K}}\to\wp(\mathcal{V}(\mathsf{At})\times\mathcal{V}(\mathsf{At}))$ that is faithful s.t.:

$$\mathcal{V}(D\star\psi)=\min_{\preceq_\phi}(\mathcal{V}(\psi)) \tag{10}$$

Appendix: Additional Proofs of Results in the Paper

Proof that $\preceq^{\mathsf{prf,d+i}}$ is a total preorder

To show that $\preceq^{\mathsf{prf,d+i}}$ is a total preorder (see Example 14), we first make some general observations on lexicographic combinations of total preorders:

Definition 26. Where $\{\preceq_i \mid 1\leq i\leq n\}$ is a set of total preorders over a set Δ, $\trianglelefteq_{i=1}^n$ is the preorder obtained as follows (where $a,b\in\Delta$): $a\trianglelefteq_{i=1}^n b$ iff there exists a $1\leq i\leq n$ s.t. $a\approx_j b$ for every $j<i$ and $a\preceq_i b$.

Proposition 28. Where $\{\preceq_i | \ 1 \leq i \leq n\}$ is a set of total preorders over Δ, $\trianglelefteq_{i=1}^n$ is reflexive, transitive and total.

Proof. Let $\{\preceq_i | \ 1 \leq i \leq n\}$ is a set of total preorders over Δ. We show that $\trianglelefteq_{i=1}^n$ is reflexive, transitive and total:

Reflexive Since $a \preceq_1 a$ for any $a \in \Delta$, we obtain $a \trianglelefteq_{i=1}^n a$ for any $a \in \Delta$.

Transitive Suppose $a, b, c \in \Delta$, $a \trianglelefteq_{i=1}^n b$ and $b \trianglelefteq_{i=1}^n c$, i.e. there are some $i_1, i_2 \in \{1, \ldots, n\}$ s.t. $a \approx_{j_1} b$ for every $j_1 < i_1$, $a \preceq_{i_1} b$, $b \approx_{j_2} c$ and $b \preceq_{j_2} c$. The case for $i_1 = j_1$ is clear. Suppose now $i_1 < i_2$. Then $b \approx_{i_1} c$ and thus by transitivity of $\preceq_{i_1}$, $a \preceq_{i_1} c$. Since $a \approx_j b$ and $b \approx_j c$ for every $1 \leq j \leq i_1$, we see that $a \trianglelefteq_{i=1}^n$. Suppose now that $i_2 < i_1$. Then $a \approx_{i_2} b$ and since $b \preceq_{i_2} c$, by transitivity of $\preceq_{i_2}$, $a \preceq_{i_2} c$. Since $a \approx_j b$ and $b \approx_j c$ for any $j < i_2$, by transitivity of $\approx_j$ (for any $j < i_2$), we obtain $a \approx_j c$ for any $j < i_2$. Altogether this implies $a \trianglelefteq_{i=1}^n$.

Total Suppose $a, b \in \Delta$. Since $\preceq_i$ is total for every $1 \leq i \leq n$, $a \approx_i b$ or $a \preceq_i b$ or $b \preceq_i a$ for every $1 \leq i \leq n$. Take $j \in \{1, \ldots, n\}$ minimal s.t. $a \not\approx_j b$. Suppose first $a \preceq_i b$. Then $a \trianglelefteq_{i=1}^n b$. Otherwise (i.e. if $b \preceq_i a$) $b \trianglelefteq_{i=1}^n a$. □

We now show that show that $\preceq^{\mathsf{prf,d+i}}$ is a total preorder:

Proposition 29. Let a set of atoms At and an ADF $D \in \mathfrak{D}(\mathsf{At})$ be given. Then $\preceq_D^{\mathsf{prf,d+i}}$ is a total preorder.

Proof. We show this by showing that $\preceq^{\mathsf{prf,d+i}}$ can be viewed as a lexicographic combination $\trianglelefteq_{i=1}^3$ (as per Definition 26), with:

- $v_1 \preceq_1 v_2$ iff $v_1 \in \mathsf{prf}(D)$;
- $v_1 \preceq_2 v_2$ iff $\mathsf{und}(v_1) \leq \mathsf{und}(v_2)$;
- $v_1 \preceq_3 v_2$ iff $\min_{v \in \mathsf{prf}(D)}(v \Delta v_1) \leq \min_{v \in \mathsf{prf}(D)}(v \Delta v_2)$.

Since $\preceq_i$ is a total preorder for $1 \leq i \leq 3$, with Proposition 28, we obtain that $\trianglelefteq_{i=1}^3$ is a total preorder.

That $\preceq^{\mathsf{prf,d+i}} = \trianglelefteq_{i=1}^3$ is immediate. □