Interpreting Conditionals in Argumentative Environments

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Abstract

In the field of knowledge representation and reasoning, different paradigms have co-existed for many years. Two central such paradigms are *conditional logics* and *formal argumentation*. Despite recent intensified efforts, the gap between these two approaches has not been fully bridged yet. In this paper, we contribute to the bridging of this gap by showing how plausible conditionals can be interpreted in argumentative reasoning enviroments. In more detail, we provide interpretations of conditional knowledge bases in *abstract dialectical frameworks*, one of the most general approaches to computational models of argumentation. We motivate the design choices made in our translation, show that different semantics give rise to several forms of adequacy, and show several desirable properties of our translation.

1 Introduction

Different paradigms of modelling human-like reasoning behaviour have emerged over the years within the field of Knowledge Representation and Reasoning. For one, *conditional logics* (Kraus, Lehmann, and Magidor 1990; Nute 1984) are a classical approach to non-monotonic reasoning that focus on the role of defeasible rules of the form $(\phi|\psi)$ with the intuitive interpretation "if ψ is true then, usually, ϕ is true as well". There exist several sophisticated reasoning approaches (Goldszmidt and Pearl 1996; Kern-Isberner 2001) that aim at resolving issues pertaining to contradictory rules. On the other hand, the more recent *argumentative approaches* (Atkinson et al. 2017) focus on the role of arguments, i. e., derivations of claims involving multiple rules, and how to resolve issues between arguments with contradictory claims. In particular, the abstract approach to formal argumentation (Dung 1995) has gained quite some interest in the wider community. One of the most general and expressive formalisms to abstract argumentation are *Abstract Dialectical Frameworks* (ADFs) (Brewka et al. 2013), which model the acceptability of arguments via general *acceptability functions*.

In this paper we investigate the correspondence between abstract dialectical frameworks and conditional logics. Syntactically, both frameworks focus on pairs of objects such as (ϕ, ψ). In conditional logic, these pairs are interpreted as conditionals with the informal meaning "if ϕ is true then, usually, ψ is true as well" and written as $(\psi|\phi)$. In abstract dialectical frameworks, these pairs are interpreted as acceptance conditions, and interpreted as "if ϕ is accepted then ψ is accepted as well". The resemblance of these informal interpretations is striking, but both approaches use fundamentally different semantics to formalise these interpretations.

In previous works (Kern-Isberner and Thimm 2018; Heyninck, Kern-Isberner, and Thimm 2020) we looked at the question of what happens if we translate an ADF into a conditional logic knowledge base, used conditional logic reasoning mechanisms on the latter, and interpreted the results in argumentative terms. Our results showed, that the intuition behind the semantics of the two worlds is generally different, but there are also cases where their semantics coincide. In this paper, we look at the complementary question from before. We investigate what happens if we translate a conditional logic knowledge base into an ADF, use ADF reasoning mechanisms on the latter, and interpret the results in conditional logic terms.

Outline of this Paper: After introducing the necessary preliminaries in Section 2 on propositional logic (Section 2.1), conditional logic (Section 2.2) and abstract dialectial frameworks (Section 2.3), we present our argumentative interpretation of conditionals in Section 3. We first present our translation for literal conditional knowledge bases (Section 3.2) and discuss the behaviour of the negation needed in this translation (Section 3.3). Thereafter we show the adequacy of this translation under both two-valued semantics in Section 3.4 and under other semantics in Section 3.5. We then generalize the translation as to allow for what we call *extended literal conditional knowledge bases* (Section 3.6) and discuss several properties of our translation in Section 3.7. Thereafter, we further motivate the design choices made in our interpretation in Section 4. Finally, we compare our work with related work (Section 5) and conclude in Section 6.

2 Preliminaries

In the following, we briefly recall some general preliminaries on propositional logic, as well as technical details on conditional logic and ADFs (Brewka et al. 2013).

2.1 Propositional Logic

For a set At of atoms let $\mathcal{L}(\mathsf{At})$ be the corresponding propositional language constructed using the usual connectives $\wedge$ (*and*), $\vee$ (*or*), $\neg$ (*negation*) and $\rightarrow$ (*material implication*). We will sometimes write $\mathring{\phi}$ to denote some element of $\{\phi, \neg\phi\}$. The set of literals is denoted by $\mathsf{Lit} = \{\mathring{\phi} \mid \phi \in \mathsf{At}\}$. A (classical) *interpretation* (also called *possible world*) ω for a propositional language $\mathcal{L}(\mathsf{At})$ is a function $\omega : \mathsf{At} \rightarrow \{\top, \bot\}$. Let $\Omega(\mathsf{At})$ denote the set of all interpretations for At. We simply write Ω if the set of atoms is implicitly given. An interpretation ω *satisfies* (or is a *model* of) an atom $a \in \mathsf{At}$, denoted by $\omega \models a$, if and only if $\omega(a) = \top$. The satisfaction relation $\models$ is extended to formulas as usual. As an abbreviation we sometimes identify an interpretation ω with its *complete conjunction*, i. e., if $a_1, \ldots, a_n \in \mathsf{At}$ are those atoms that are assigned $\top$ by ω and $a_{n+1}, \ldots, a_m \in \mathsf{At}$ are those propositions that are assigned $\bot$ by ω we identify ω by $a_1 \ldots a_n \overline{a_{n+1}} \ldots \overline{a_m}$ (or any permutation of this). For example, the interpretation ω_1 on $\{a, b, c\}$ with $\omega(a) = \omega(c) = \top$ and $\omega(b) = \bot$ is abbreviated by $a\bar{b}c$. For $\Phi \subseteq \mathcal{L}(\mathsf{At})$ we also define $\omega \models \Phi$ if and only if $\omega \models \phi$ for every $\phi \in \Phi$. Define the set of models $\mathsf{Mod}(X) = \{\omega \in \Omega(\mathsf{At}) \mid \omega \models X\}$ for every formula or set of formulas X. A formula or set of formulas X_1 *entails* another formula or set of formulas X_2, denoted by $X_1 \vdash X_2$, if $\mathsf{Mod}(X_1) \subseteq \mathsf{Mod}(X_2)$.

2.2 Reasoning with Nonmonotonic Conditionals

Conditional logics are concerned with conditionals of the form $(\phi|\psi)$ whose informal meaning is "if ψ is true then, usually, ϕ is true as well". A *conditional knowledge base* Δ is a set of such conditionals. It is *atomic* if for every $(\phi|\psi) \in \Delta$, $\phi, \psi \in \mathsf{At}$ and it is literal if for every $(\phi|\psi) \in \Delta$, $\phi, \psi \in \mathsf{Lit}$. We will not count the constants $\top$ or $\bot$ as atoms or literals. If for every $(\phi|\psi) \in \Delta$, $\phi, \psi \in \mathsf{Lit} \cup \{\top\}$, we say Δ is an *extended literal conditional knowledge base*. There are many different conditional logics (cf., e. g., (Kraus, Lehmann, and Magidor 1990; Nute 1984)), and we will just use basic properties of conditionals that are common to many conditional logics and are especially important for nonmonotonic reasoning: Basically, we follow the approach of de Finetti (de Finetti 1974) who considered conditionals as *generalized indicator functions* for possible worlds resp. propositional interpretations ω:

$$((\psi|\phi))(\omega) = \begin{cases} 1 & : \quad \omega \models \phi \wedge \psi \\ 0 & : \quad \omega \models \phi \wedge \neg\psi \\ u & : \quad \omega \models \neg\phi \end{cases} \qquad (1)$$

where u stands for *unknown* or *indeterminate*. In other words, a possible world ω *verifies* a conditional $(\psi|\phi)$ iff it satisfies both antecedent and conclusion ($(\psi|\phi)(\omega) = 1$); it *falsifies, or violates* it iff it satisfies the antecedence but not the conclusion ($(\psi|\phi)(\omega) = 0$); otherwise the conditional is *not applicable*, i. e., the interpretation does not satisfy the antecedence ($(\psi|\phi)(\omega) = u$). We say that ω *satisfies* a conditional $(\psi|\phi)$ iff it does not falsify it, i. e., iff ω satisfies its *material counterpart* $\phi \rightarrow \psi$. Hence, conditionals are three-valued logical entities and thus extend the binary setting of classical logics substantially in a way that is compatible with the probabilistic interpretation of conditionals as conditional probabilities. Such a conditional $(\psi|\phi)$ can be accepted as plausible if its verification $\phi \wedge \psi$ is more plausible than its falsification $\phi \wedge \neg\psi$, where plausibility is often modelled by a total preorder on possible worlds. This is in full compliance with nonmonotonic inference relations $\phi \mid\!\sim \psi$ (Makinson 1988) expressing that from ϕ, ψ may be plausibly/defeasibly derived. An obvious implementation of total preorders are *ordinal conditional functions (OCFs)*, (also called *ranking functions*) $\kappa : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$ (Spohn 1988). They express degrees of (im)plausibility of possible worlds and propositional formulas ϕ by setting $\kappa(\phi) := \min\{\kappa(\omega) \mid \omega \models \phi\}$. OCFs κ provide a particularly convenient formal environment for nonmonotonic and conditional reasoning, allowing for simply expressing the acceptance of conditionals and nonmonotonic inferences via stating that $(\psi|\phi)$ is accepted by κ iff $\phi \mid\!\sim_\kappa \psi$ iff $\kappa(\phi \wedge \psi) < \kappa(\phi \wedge \neg\psi)$, implementing formally the intuition of conditional acceptance based on plausibility mentioned above. For an OCF κ, *Bel* (κ) denotes the propositional beliefs that are implied by all most plausible worlds, i. e. *Bel* $(\kappa) = \{\phi \mid \forall \omega \in \kappa^{-1}(0) : \omega \models \phi\}$. We write $\kappa \models \phi$ if $\phi \in$ *Bel* (κ).

Specific examples of ranking models are system Z yielding the inference relation $\mid\!\sim_{\mathrm{Z}}$ (Goldszmidt and Pearl 1996) and c-representations (Kern-Isberner 2001). We discuss system Z defined as follows. A conditional $(\psi|\phi)$ is tolerated by a finite set of conditionals Δ if there is a possible world ω with $(\psi|\phi)(\omega) = 1$ and $(\psi'|\phi')(\omega) \neq 0$ for all $(\psi'|\phi') \in \Delta$, i. e. ω verifies $(\psi|\phi)$ and does not falsify any (other) conditional in Δ. The Z-partitioning $(\Delta_0, \ldots, \Delta_n)$ of Δ is defined as:

- $\Delta_0 = \{\delta \in \Delta \mid \Delta \text{ tolerates } \delta\}$;
- $\Delta_1, \ldots, \Delta_n$ is the Z-partitioning of $\Delta \setminus \Delta_0$.

For $\delta \in \Delta$ we define: $Z_\Delta(\delta) = i$ iff $\delta \in \Delta_i$ and $(\Delta_0, \ldots, \Delta_n)$ is the Z-partioning of Δ. Finally, the ranking function κ^Z_Δ is defined via: $\kappa^Z_\Delta(\omega) = \max\{Z(\delta) \mid \delta(\omega) = 0, \delta \in \Delta\} + 1$, with $\max \emptyset = -1$. We can now define $\Delta \mid\!\sim_{\mathrm{Z}} \phi$ iff $\top \mid\!\sim_{\kappa^Z_\Delta} \phi$ (which can be seen to be equivalent to $\phi \in$ *Bel* (κ^Z_Δ)).

Below the following Lemma about system Z will prove useful:

Lemma 1. *Let $\omega \in \Omega$ and Δ be a conditional knowledge base. Then $\omega \notin (\kappa^Z_\Delta)^{-1}(0)$ iff $\delta(\omega) = 0$ for some $\delta \in \Delta$.*

Proof. This follows immediately in view of the fact that $\omega \in (\kappa^Z_\Delta)^{-1}(0)$ iff $\delta(\omega) \neq 0$ for every $\delta \in \Delta$. $\square$

We now illustrate OCFs in general and System Z in particular with the well-known "Tweety the penguin"-example.

Example 1. *Let $\Delta = \{(f|b), (b|p), (\neg f|p)\}$, which expresses that most birds (b) fly (f), most penguins ((p)) are birds, and most penguins do not fly. This conditional knowledge base has the following Z-partitioning: $\Delta_0 = \{(f|b)\}$ and $\Delta_1 = \{(b|p), (\neg f|p)\}$. This gives rise to the following κ^Z_Δ-ordering over the worlds based on the signature $\{b, f, p\}$:*

ω	κ^Z_Δ	ω	κ^Z_Δ	ω	κ^Z_Δ	ω	κ^Z_Δ
bpf	2	$bp\overline{f}$	1	$b\overline{p}f$	0	$b\overline{p}\overline{f}$	1
$\overline{b}pf$	2	$\overline{b}p\overline{f}$	2	$\overline{b}\overline{p}f$	0	$\overline{b}\overline{p}\overline{f}$	0

As an example of a κ^Z_Δ-belief, observe that $\neg p, \neg(b \wedge \neg f) \in Bel(\kappa^Z_\Delta)$.

2.3 Abstract Dialectical Frameworks

We briefly recall some technical details on abstract dialectical frameworks (ADF) following loosely the notation from (Brewka et al. 2013). We can depict an ADF D as a directed graph whose nodes represent statements or arguments which can be accepted or not. With links we represent dependencies between nodes. A node s is depended on the status of the nodes with a direct link to s, denoted parent nodes $par_D(s)$. With an acceptance function C_s we define the cases when the statement s can be accepted (truth value $\top$), depending on the acceptance status of its parents in D.

An ADF D is a tuple $D = (S, L, C)$ where S is a set of statements, $L \subseteq S \times S$ is a set of links, and $C = \{C_s\}_{s \in S}$ is a set of total functions $C_s : 2^{par_D(s)} \to \{\top, \bot\}$ for each $s \in S$ with $par_D(s) = \{s' \in S \mid (s', s) \in L\}$. By abuse of notation, we will often identify an acceptance function C_s by its equivalent *acceptance condition* which models the acceptable cases as a propositional formula.

An ADF $D = (S, L, C)$ is interpreted through 3-valued interpretations $v : S \to \{\top, \bot, u\}$, which assign to each statement in S either the value $\top$ (true, accepted), $\bot$ (false, rejected), or u (unknown).

A 3-valued interpretation v can be extended to arbitrary propositional formulas over S via strong Kleene semantics:

1. $v(\neg\phi) = \bot$ iff $v(\phi) = \top$, $v(\neg\phi) = \top$ iff $v(\phi) = \bot$, and $v(\neg\phi) = u$ iff $v(\phi) = u$;
2. $v(\phi \wedge \psi) = \top$ iff $v(\phi) = c(\psi) = \top$, $v(\phi \wedge \psi) = \bot$ iff $v(\phi) = \bot$ or $v(\psi) = \bot$, and $v(\phi \wedge \psi) = u$ otherwise;
3. $v(\phi \vee \psi) = \top$ iff $v(\phi) = \top$ or $v(\psi) = \top$, $v(\phi \vee \psi) = \bot$ iff $v(\phi) = c(\psi) = \bot$, and $v(\phi \vee \psi) = u$ otherwise.

$\mathcal{V}$ consists of all three-valued interpretations whereas $\mathcal{V}^2$ consists of all the two-valued interpretations (i. e. interpretations such that for every $s \in S$, $v(s) \in \{\top, \bot\}$). Then v is a *model* of D if for all $s \in S$, if $v(s) \neq u$ then $v(s) = v(C_s)$.

We define an order $\leq_i$ over $\{\top, \bot, u\}$ by making u the minimal element: $u <_i \top$ and $u <_i \bot$ and this order is lifted pointwise as follows (given two valuations v, w over S): $v \leq_i w$ iff $v(s) \leq_i w(s)$ for every $s \in S$. So intuitively the classical truth values contain more information than the truth value u. The set of two-valued interpretations extending a valuation v is defined as $[v]^2 = \{w \in \mathcal{V}^2 \mid v \leq_i w\}$. Given a set of valuations V, $\sqcap_i V(s) = v(s)$ if for every $v' \in V$, $v(s) = v'(s)$ and $\sqcap_i V(s) = u$ otherwise. $\Gamma_D(v) : S \to \{\top, \bot, u\}$ where $s \mapsto \sqcap_i\{w(C_s) \mid w \in [v]^2\}$.

For the definition of the stable model semantics, we need to define the reduct D^v of D given v, defined as: $D^v = (S^v, L^v, C^v)$ with:

- $S^v = \{s \in S \mid v(s) = \top\}$,
- $L^v = L \cap (S^v \times S^v)$, and
- $C^v = \{C_s[\{\phi \mid v(\phi) = \bot\}/\bot] \mid s \in S^v\}$.

where $C_s[\phi/\psi]$ is the formula obtained by substituting every occurence of ϕ in C_s by ψ.

Definition 1. *Let $D = (S, L, C)$ be an ADF with $v : S \to \{\top, \bot, u\}$ an interpretation:*

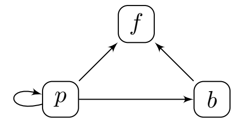

Figure 1: Graph representing links between nodes for D in Example 2.

- *v is a 2-valued model iff $v \in \mathcal{V}^2$ and v is a model.*
- *v is complete for D iff $v = \Gamma_D(v)$.*
- *v is preferred for D iff v is $\leq_i$-maximally complete for D.*
- *v is grounded for D iff v is $\leq_i$-minimally complete for D.*
- *v is stable iff v is a model of D and $\{s \in S \mid v(s) = \top\} = \{s \in S \mid w(s) = \top\}$ where w is the grounded interpretation of D^v.*

We denote by $\mathsf{2mod}(D)$, $\mathsf{complete}(D)$, $\mathsf{preferred}(D)$ respectively $\mathsf{stable}(D)$ the sets of 2-valued models and complete, preferred, respectively stable interpretations of D. The grounded interpretation, which in (Brewka and Woltran 2010) is shown to be unique, will be denoted by v^D_{g}. If D is clear from the context we will just write v_{g}.

Notice that any complete interpretation is also a model.

We finally define consequence relations for ADFs:

Definition 2. *Given $\mathsf{sem} \in \{\mathsf{2mod}, \mathsf{preferred}, \mathsf{stable}\}$, an ADF $D = (S, L, C)$ and $s \in \mathcal{L}(S)$, we define: $D \mathrel{|\!\sim}^{\cap}_{\mathsf{sem}} s[\neg s]$ iff $v(s) = \top[\bot]$ for all $v \in \mathsf{sem}(D)$. $D \mathrel{|\!\sim}_{\mathsf{grounded}} s[\neg s]$ iff $v^D_{\mathsf{g}}(s) = \top[\bot]$.*

We illustrate ADFs by looking at a naive formalization of the Penguin-example in abstract dialectical argumentation:

Example 2. *Let $D = (\{p, b, f\}, L, C)$ with $C_p = p$, $C_b = p$ and $C_f = \neg p \vee b$. The corresponding graph for D can be find in Figure 1. This ADF has two two-valued models, which are also its preferred models: v_1 with $v_1(p) = v_1(b) = \bot$ and $v_1(f) = \top$ and v_2 with $v_2(p) = v_2(b) = v_2(f) = \top$. The grounded interpretation assigns u to all nodes p, b and f.*

3 Interpreting Conditionals in ADFs

In (Heyninck, Kern-Isberner, and Thimm 2020) we looked at the problem of translating an ADF into a conditional logic knowledge base. We now look at the complementary question, namely translating a conditional logic knowledge base into an ADF. These two translations will help to better understand the connection between argumentation and reasoning from conditional knowledge bases.

In this section, we present an interpretation of conditional knowledge bases into abstract dialectical frameworks. In Section 3.1 we introduce the language used for translating knowledge bases and formulate several notions of adequacy used for evaluating our translation. The translation is presented in Section 3. In Section 3.3 we discuss the use of the newly introduced negation, whereafter we show the adequacy of our translation under two-valued (Section 3.4) and other semantics (Section 3.5). Thereafter, we discuss how to translate normality statements in Section 3.6 and finally we discuss properties of the translation in Section 3.7.

3.1 Translations of Conditionals into ADFs

To obtain an adequate translation, it will prove useful to extend the language with a new atomic negation operator $\widetilde{\ }$. We denote the set of atoms negated by this new negation by $\widetilde{\mathsf{At}} = \{\widetilde{\phi} \mid \phi \in \mathsf{At}\}$. $\mathsf{Lit}\text{~}(\mathsf{At}) = \mathsf{At} \cup \widetilde{\mathsf{At}}$. When At is clear from the context, we will somtimes just write $\mathsf{Lit}\text{~}$. It will prove useful to define the following notions:

Definition 3. *We define the functions*

$$\ulcorner\cdot\urcorner : \mathsf{Lit} \to \mathsf{Lit}\text{~}$$
$$\llcorner\cdot\lrcorner : \mathsf{Lit}\text{~} \to \mathsf{Lit}$$
$$-\cdot : \mathsf{Lit}\text{~} \to \mathsf{Lit}\text{~}$$

with:

$$\ulcorner\phi\urcorner = \begin{cases} \phi & \textit{if } \phi \in \mathsf{At} \\ \widetilde{\psi} & \textit{if } \phi = \neg\psi \textit{ for some } \psi \in \mathsf{At} \end{cases}$$

$$\llcorner\phi\lrcorner = \begin{cases} \phi & \textit{if } \phi \in \mathsf{At} \\ \neg\psi & \textit{if } \phi = \widetilde{\psi} \textit{ for some } \psi \in \mathsf{At} \end{cases}$$

$$-\phi = \begin{cases} \widetilde{\phi} & \textit{if } \phi \in \mathsf{At} \\ \psi & \textit{if } \phi = \widetilde{\psi} \textit{ for some } \psi \in \mathsf{At} \end{cases}$$

Let $\mathfrak{C}^{\mathsf{li}}(\mathsf{At})$ is the set of all literal conditional knowledge bases over At and $\mathfrak{D}(\mathsf{Lit}\text{~}(\mathsf{At}))$ all the ADFs defined on the basis of S (i. e. $D = (\mathsf{Lit}\text{~}(\mathsf{At}), L, C)$). In this paper, we consider translations $D : \mathfrak{C}^{\mathsf{li}}(\mathsf{At}) \to \mathfrak{D}(\mathsf{Lit}\text{~}(\mathsf{At}))$, and in particular translations which preserve the meaning of the translated knowledge base Δ. In more detail, we will use two notions of adequacy to evaluate translations.

The first notion is *respecting* Δ and is based on de Finetti's conception of conditionals as generalized indicator functions to worlds described above. Indeed, given a conditional knowledge base Δ we can straightforwardly extend (de Finetti 1974)'s notion of conditionals as generalized indicator functions to worlds in $\Omega(\mathsf{Lit}\text{~}(\mathsf{At}(\Delta)))$. In more detail, for such an $\omega \in \Omega(\mathsf{Lit}\text{~}(\mathsf{At}(\Delta)))$, we define:

$$((\psi|\phi))(\omega) = \begin{cases} 1 : & \omega \models \ulcorner\phi\urcorner \wedge \ulcorner\psi\urcorner \\ u : & \omega \models \neg\ulcorner\phi\urcorner \\ 0 : & \omega \models \ulcorner\phi\urcorner \wedge -\ulcorner\psi\urcorner \end{cases}$$

We will say that an interpretation $\omega \in \Omega(\mathsf{Lit}\text{~}(\mathsf{At}(\Delta)))$ *respects* Δ if $(\delta)(\omega) \neq 0$ for any $\delta \in \Delta$.

The second notion of adequacy is stronger and requires equivalence on the level of the non-monotonic inference relation. In more detail, we say a translation D is *inferentially equivalent w.r.t. an ADF-based inference relation*[1] $\mid\!\sim$ if for any conditional knowledge base Δ: $\Delta \mid\!\sim_{\mathsf{Z}} \phi$ iff $D(\Delta) \mid\!\sim \phi$. Clearly, inferential equivalence w.r.t. $\mid\!\sim_{\mathsf{sem}}$ (for some semantics sem) of a translation $D : \mathfrak{C}^{\mathsf{li}}(\mathsf{At}) \to \mathfrak{D}(\mathsf{Lit}\text{~}(\mathsf{At}))$ implies that all the interpretations in $\mathsf{sem}(D(\Delta))$ respect Δ for any literal conditional knowledge base Δ.

[1] An ADF-based infrence relation is a relation $\mid\!\sim\ \subseteq \mathfrak{D}(S) \times \mathcal{L}(S)$. Examples of such inference relations are those defined in Definition 2.

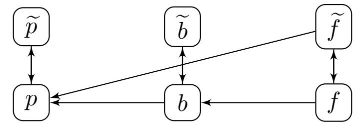

Figure 2: Graph representing the links between nodes of $D_1(\Delta)$ in Example 3.

3.2 Translation D_1

The guiding idea behind our first translation is that given a conditional $(p|q)$, what we take into account is the following behaviour: if q is believed then p should be believed. Now one way to translate this in ADFs is to have q as a positive or a supporting link for p. Another way to formalize this idea, however, is to require that q can be believed only if so is p, i. e. $\{C_q\} \vdash p$. In other words the consequent p is a supporting link of the antecedent q. We will here explore the latter idea and show in Section 4.2 that the former idea leads to inadequate translations.

We are now ready to define our translation D_1 from conditional knowledge bases into ADFs.

Definition 4. *Given a literal conditional knowledge base Δ, we define: $D_1(\Delta) = (\mathsf{Lit}\text{~}(\mathsf{At}(\Delta)), L, C)$ where: $C_\phi = \neg - \phi \wedge \bigwedge_{(\psi|\llcorner\phi\lrcorner)\in\Delta} \ulcorner\psi\urcorner$ for any $\phi \in \{\psi, \widetilde{\psi} \mid \psi \in \mathsf{At}(\Delta)\}$.*

Given a literal $\phi \in \mathsf{Lit}\text{~}(\mathsf{At}(\Delta))$, the intuition behind C_ϕ is the following. The first part $\neg - \phi$ ensures that $\widetilde{\ }$ behaves like a negation by ensuring that the contrary $-\phi$ of ϕ is not believed when ϕ is believed. The second part of the condition C_ϕ, $\bigwedge_{(\psi|\llcorner\phi\lrcorner)\in\Delta} \ulcorner\psi\urcorner$, ensures that conditionals are interpreted adequately. In more detail, it ensures that ϕ is only believed if for every conditional $(\psi|\llcorner\phi\lrcorner)$ which has ϕ as an antecedent (modulo transformation to the original language Lit), the consequent $\ulcorner\psi\urcorner$ is believed (again, modulo transformation into the extended language $\mathsf{Lit}\text{~}$) .

Notice that for any $\phi \in \mathsf{At}$, the conditions can be equivalently written as (where ψ is an atom):

- $C_\phi = \neg\widetilde{\phi} \wedge \bigwedge_{(\mathring{\psi}|\phi)\in\Delta} \ulcorner\mathring{\psi}\urcorner$.
- $C_{\widetilde{\phi}} = \neg\phi \wedge \bigwedge_{(\mathring{\psi}|\neg\phi)\in\Delta} \ulcorner\mathring{\psi}\urcorner$.

We illustrate our translation by first looking at the Tweety-example:

Example 3. $\Delta = \{(f|b), (b|p), (\neg f|p)\}$. *The following nodes are part of the ADF: $\{b, \widetilde{b}, f, \widetilde{f}, p, \widetilde{p}\}$. We have the following conditions:*

- $C_b = \neg\widetilde{b} \wedge f$
- $C_p = \neg\widetilde{p} \wedge b \wedge \widetilde{f}$
- $C_x = \neg - x$ *for* $x \in \{f, \widetilde{f}, \widetilde{b}, \widetilde{p}\}$.

The corresponding graph can be found in Figure 2.

We can read this as follows: b can be believed whenever it is not believed that $\widetilde{b}$ (i. e. nothing is both a bird and a not-bird) and it is believed that f (i. e. something is a bird only if it flies). Argumentatively, $\widetilde{b}$ attacks b and f supports b. Likewise, b and $\widetilde{f}$ support p (whereas $\widetilde{p}$ attacks p).

$D_1(\Delta)$ has the following two-valued models:

i	$v_i(b)$	$v_i(\widetilde{b})$	$v_i(f)$	$v_i(\widetilde{f})$	$v_i(p)$	$v_i(\widetilde{p})$
1	$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$
2	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$
3	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$

Notice that these two-valued models correspond to the most plausible worls according to κ^Z_Δ (see Example 1).

Another benchmark example well-known from the literature is the so-called *Nixon diamond*, where equally plausible rules lead to mutually inconsistent conclusions.

Example 4 (The Nixon Diamond)**.** *Let $\Delta = \{(p|q), (\neg p|r)\}$. Then $D_1(\Delta) = (\{p, \widetilde{p}, q, \widetilde{q}\}, L, C)$ with:*

- $C_q = \neg\widetilde{q} \wedge p$
- $C_r = \neg\widetilde{r} \wedge \widetilde{p}$
- $C_x = \neg - x$ *for* $x \in \{p, \widetilde{p}, \widetilde{q}, \widetilde{r}\}$

$2\mathsf{mod}(D_1(\Delta)) = \{v_1, v_2, v_3, v_4\}$ *with:*

i	$v_i(q)$	$v_i(\widetilde{q})$	$v_i(r)$	$v_i(\widetilde{r})$	$v_i(p)$	$v_i(\widetilde{p})$
1	$\top$	$\bot$	$\bot$	$\top$	$\top$	$\bot$
2	$\bot$	$\top$	$\top$	$\bot$	$\bot$	$\top$
3	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$
4	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$

It can be observed that $(\kappa^Z_\Delta)^{-1}(0) = \{pq\overline{r}, \overline{pq}r, \mathring{p}\overline{qr}\}$. As in the previous example, $2\mathsf{mod}(D_1(\Delta))$ corresponds to $(\kappa^Z_\Delta)^{-1}(0)$.

In the Section 3.4, we will see that the correspondence between $2\mathsf{mod}(D_1(\Delta))$ and $(\kappa^Z_\Delta)^{-1}(0)$ in the above examples is no coincidence.

3.3 Properties of $\widetilde{\ }$

Before discussing the adequacy of the translation Δ_1, it is important to ask whether $\widetilde{\ }$ fulfills some well-known properties of negations, such as *completeness* and *consistency*. Completeness of $\widetilde{\ }$ in an interpretation ω means that for every $\phi \in \mathsf{At}$, at least one of ϕ and $\widetilde{\phi}$ is true in ω, whereas consistency in an interpretation ω means that at most one of ϕ and $\widetilde{\phi}$ is true in ω (for any $\phi \in \mathsf{At}$).

Definition 5. *Given $\omega \in \Omega(\mathsf{At} \cup \widetilde{\mathsf{At}})$, we say $\widetilde{\ }$ is:*

- complete in ω *if for all* $\phi \in \mathsf{At}$, $\omega(\phi) = \top$ *or* $\omega(\widetilde{\phi}) = \top$.
- consistent in ω *if for all* $\phi \in \mathsf{At}$, $\omega(\phi) = \bot$ *or* $\omega(\widetilde{\phi}) = \bot$.

We can illustrate these definitions with a simple example:

Example 5. *Consider the following interpretations of $\{p, \widetilde{p}\}$:*

i	$v_i(p)$	$v_i(\widetilde{p})$	*is v_i consistent?*	*is v_i complete?*
1	$\bot$	$\bot$	*yes*	*no*
2	$\bot$	$\top$	*yes*	*yes*
3	$\top$	$\top$	*no*	*yes*
4	u	u	*no*	*no*

We first observe that there extist knowledge bases Δ for which there are two-valued models ω of $D_1(\Delta)$ s.t. $\widetilde{\ }$ is not complete in ω, as witnessed by the following example:

Example 6. $\Delta = \{(\overline{p}|q), (\overline{p}|\overline{q})\}$. *We have $D(\Delta) = (\{p, q, \widetilde{p}, \widetilde{q}\}, L, C)$ with $C_q = \neg\widetilde{q} \wedge \widetilde{p}$, $C_{\widetilde{q}} = \neg q \wedge \widetilde{p}$, $C_p = \neg\widetilde{p}$, $C_{\widetilde{p}} = \neg p$. This ADF has the following two-valued models:*

i	$v_i(p)$	$v_i(\widetilde{p})$	$v_i(q)$	$v_i(\widetilde{q})$
1	$\bot$	$\top$	$\bot$	$\top$
2	$\bot$	$\top$	$\top$	$\bot$
3	$\top$	$\bot$	$\bot$	$\bot$

Notice that v_3 is a two-valued model since $v_3(\neg\widetilde{p}) = \bot$ and thus $v_3(C_q) = v_1(C_{\widetilde{q}}) = \bot$. This two-valued model interprets $\widetilde{\ }$ as an incomplete negation (i. e. there might be $\widetilde{\ }$-gaps), since both q and $\widetilde{q}$ are false in v_3.

However, for any literal knowledge base Δ and any two-valued model ω of $D_1(\Delta)$, $\widetilde{\ }$ is a consistent in ω (i.e. there are no $\widetilde{\ }$-gluts):

Proposition 1. *Let a literal conditional knowledge base Δ, some $\phi \in \mathsf{At}(\Delta)$, and $\omega \in 2\mathsf{mod}(D_1(\Delta))$ be given. Then $\omega(\phi) = \top$ implies $\omega(\widetilde{\phi}) = \bot$ and $\omega(\widetilde{\phi}) = \top$ implies $\omega(\phi) = \bot$.*

Proof. Suppose Δ is a literal conditional knowledge base and $\phi \in \mathsf{At}(\Delta)$ and $\omega \in 2\mathsf{mod}(D_1(\Delta))$. Suppose now $\omega(\phi) = \top$. Since $\omega \in 2\mathsf{mod}(D(\Delta))$, $\omega(\phi) = \omega(C_\phi)$. Since $C_\phi = \neg\widetilde{\phi} \wedge \bigwedge_{(\psi|\phi)\in\Delta} \ulcorner\psi\urcorner$, $\omega(C_\phi) = \top$ implies $\omega(\neg\widetilde{\phi}) = \top$, i. e. $\omega(\widetilde{\phi}) = \bot$. The case for $\omega(\widetilde{\phi})$ is analogous. □

3.4 Adequacy of Translation D_1

We first show that two-valued models of $D_1(\Delta)$ respect Δ:

Proposition 2. *Let a literal conditional knowledge base Δ, $\omega \in 2\mathsf{mod}(D_1(\Delta))$ and $(\phi|\psi) \in \Delta$ be given. Then $\omega(\ulcorner\psi\urcorner) = \top$ implies $\omega(\ulcorner\phi\urcorner) = \top$.*

Proof. Suppose that $\omega \in 2\mathsf{mod}(D(\Delta))$ and let $(\phi|\psi) \in \Delta$. Suppose that $\omega(\ulcorner\psi\urcorner) = \top$. We assume first that $\psi, \phi \in \mathsf{At}$. Since $C_\psi = \neg\widetilde{\psi} \wedge \bigwedge_{(\phi'|\psi)\in\Delta} \ulcorner\phi'\urcorner$ and $(\phi|\psi) \in \Delta$,

$$C_\psi = \neg\widetilde{\psi} \wedge \phi \wedge \bigwedge_{(\phi'|\psi)\in\Delta\setminus\{(\phi|\psi)\}} \ulcorner\phi'\urcorner$$

and thus $C_\psi \vdash \phi$. Since $\omega \in 2\mathsf{mod}(D(\Delta))$, $\omega(\psi) = \omega(C_\psi) = \top$. Since $C_\psi \vdash \phi$, this means $\omega(\phi) = \top$. Since $\phi \in \mathsf{At}$, this implies $\omega(\ulcorner\phi\urcorner) = \top$. The other cases are analogous. □

Corollary 1. *Let a literal conditional knowledge base Δ be given. Then any $\omega \in 2\mathsf{mod}(D_1(\Delta))$ respects Δ.*

Proof. By Proposition 2, for any $\omega \in 2\mathsf{mod}(D_1(\Delta))$ and any $(\phi|\psi) \in \Delta$, $\omega(\ulcorner\psi\urcorner) = \bot$ or $\omega(\ulcorner\psi\urcorner \wedge \ulcorner\phi\urcorner) = \top$, which implies that $\omega((\phi|\psi)) \neq 0$. □

We can now easily show that every two-valued model of $D_1(\Delta)$ corresponds to a maximally plausible world ω. We first have to define a function that allows us to associate two-valued models in the language using $\widetilde{\ }$ with the worlds $\Omega(\mathsf{At})$ (and vice-versa).

Definition 6. *Where $\omega \in \Omega(\mathsf{Lit}^{\sim}(\mathsf{At}))$ and $\widetilde{\ }$ is complete in ω, we define $\omega{\downarrow} \in \Omega(\mathsf{At})$ as the world such that for every $\phi \in \mathsf{At}$:*

$$\omega{\downarrow}(\phi) = \begin{cases} \top & \text{if } \omega(\phi) = \top \\ \bot & \text{if } \omega(\widetilde{\phi}) = \top \end{cases}$$

Let $\omega \in \Omega(\mathsf{At})$. Then we define $\omega{\downarrow} \in \Omega(\mathsf{Lit}^{\sim})$ as the world such that for every $\phi \in \mathsf{At}$:

$$\omega{\uparrow}(\phi) = \top \text{ and } \omega{\uparrow}(\widetilde{\phi}) = \bot \text{ iff } \omega(\phi) = \top$$
$$\omega{\uparrow}(\widetilde{\phi}) = \top \text{ and } \omega{\uparrow}(\phi) = \bot \text{ iff } \omega(\phi) = \bot$$

We can now show the correspondence between $\widetilde{\ }$-complete two-valued models and maximally plausible worlds.

Proposition 3. *Let a literal conditional knowledge base Δ and an $\omega \in \mathsf{2mod}(D_1(\Delta))$ for which $\widetilde{\ }$ is complete in ω be given. Then $\kappa^Z_\Delta(\omega{\downarrow}) = 0$.*

Proof. Suppose Δ is a literal conditional knowledge base, $\omega \in \mathsf{2mod}(D_1(\Delta))$ and $\widetilde{\ }$ is complete in ω. Indeed, let $(\phi|\psi) \in \Delta$ and suppose $\omega{\downarrow} \models \ulcorner\psi\urcorner$. By Definition 6, this implies $\omega \models \psi$. With Proposition 2, this implies that $\omega \models \ulcorner\phi\urcorner$. Again with Definition 6, this implies $\omega{\downarrow} \models \phi$. Thus, we have established that if $\omega \in \mathsf{2mod}(D_1(\Delta))$ and $\widetilde{\ }$ is complete in ω then $\omega{\downarrow} \not\models \psi \wedge \neg\phi$ for any $(\phi|\psi) \in \Delta$, i.e. $((\phi|\psi))(\omega{\downarrow}) \neq 0$ (for any $(\phi|\psi) \in \Delta$). With Lemma 1 this means $\kappa^Z_\Delta(\omega) = 0$. $\square$

Fact 1. *For any $\omega \in \Omega$, $\widetilde{\ }$ is complete in $\omega{\uparrow}$.*

Lemma 2. *Let a literal conditional knowledge base Δ and some $\omega \in \Omega$ be given. Then if $\kappa^Z_\Delta(\omega) = 0$ then $\omega{\uparrow} \in \mathsf{2mod}(D_1(\Delta))$.*

Proof. Let a literal conditional knowledge base Δ and some $\omega \in \Omega$ be given. Consider some $\phi \in \mathsf{Lit}^{\sim}$. We show that $\omega{\uparrow} \models \phi$ iff $\omega{\uparrow} \models C_\phi$, which implies $\omega{\uparrow}$ is a two-valued model of $D_1(\Delta)$. For this suppose first that $\omega{\uparrow} \models \phi$ and suppose towards a contradiction $\omega{\uparrow} \models \neg C_\phi$, i.e. $\omega{\uparrow} \models \widetilde{\phi} \vee \neg \bigwedge_{(\psi|\llcorner\phi\lrcorner)\in\Delta} \ulcorner\psi\urcorner$. With Proposition 1 and since $\omega{\uparrow} \models \phi$, $\omega{\uparrow} \not\models \widetilde{\phi}$, which implies $\omega{\uparrow} \models \neg \bigwedge_{(\psi|\llcorner\phi\lrcorner)\in\Delta} \ulcorner\psi\urcorner$, i.e. there is some $(\psi|\llcorner\phi\lrcorner) \in \Delta$ s.t. $\omega{\uparrow} \models \neg\ulcorner\psi\urcorner$. By definition of $\omega{\uparrow}$, this implies that $\omega \models \neg\psi$. But then $\omega \models \phi \wedge \neg\psi$ for some $(\psi|\phi) \in \Delta$, contradiction to $\kappa^Z_\Delta(\omega) = 0$. Suppose now (again towards a contradiction) that $\omega{\uparrow} \models C_\phi$ and $\omega{\uparrow} \not\models \phi$. By Fact 1, $\omega{\uparrow} \not\models \phi$ implies $\omega{\uparrow} \models \widetilde{\phi}$. Since $C_\phi = \neg\widetilde{\phi} \wedge \bigwedge_{(\psi|\llcorner\phi\lrcorner)\in\Delta} \ulcorner\psi\urcorner$, this contradicts $\omega{\uparrow} \models C_\phi$. $\square$

Fact 2. *Let some $\omega \in \Omega(\mathsf{Lit}^{\sim})$ s.t. $\widetilde{\ }$ is complete in ω and some $\phi \in \mathsf{At}$ be given. Then $\omega \models \widetilde{\phi}$ iff $\omega \models \neg\phi$.*

Proof. Suppose first $\omega \models \widetilde{\phi}$. By Proposition 1, $\omega \not\models \phi$ and thus $\omega \models \neg\phi$. Suppose now that $\omega \models \neg\phi$. Since $\widetilde{\ }$ is complete in ω, by Definition 6, $\omega \models \widetilde{\phi}$. $\square$

Lemma 3. *Let some $\omega \in \Omega(\mathsf{Lit}^{\sim})$ s.t. $\widetilde{\ }$ is complete in ω and some $\phi \in \mathcal{L}(\mathsf{At})$ be given. Then $\omega{\downarrow} \models \phi$ iff $\omega \models \phi$.*

Proof. We show this by showing the claim for any $\phi \in \mathcal{L}(\mathsf{At})$ in *disjunctive normal form*, i.e. $\phi = \bigvee_{i=1}^n \bigwedge_{j=1}^m \mathring{\phi}_i^j$. Suppose $\omega{\downarrow} \models \phi$, i.e. there is some $1 \leq i \leq n$ s.t. $\omega{\downarrow} \models \bigwedge_{j=1}^m \mathring{\phi}_i^j$. By Fact 2 and Definition 6, this implies $\omega \models \bigwedge_{j=1}^m \mathring{\phi}_i^j$ and thus $\omega \models \bigvee_{i=1}^n \bigwedge_{j=1}^m \mathring{\phi}_i^j$. The other direction is analogous. $\square$

Given some ADF D, we define: $D \mathrel{|\!\sim}^{\cap,c}_{\mathsf{2mod}} \phi$ iff $\omega(\phi) = \top$ for every $\omega \in \mathsf{2mod}(D)$ for which $\widetilde{\ }$ is complete in ω.

Theorem 1. *Given a literal conditional knowledge base Δ, $\Delta \mathrel{|\!\sim}_Z \phi$ iff $D_1(\Delta) \mathrel{|\!\sim}^{\cap,c}_{\mathsf{2mod}} \phi$.*

Proof. Suppose first that $\Delta \mathrel{|\!\sim}_Z \phi$, i.e. for every $\omega \in \Omega$ s.t. $\kappa^Z_\Delta(\omega) = 0$, $\omega \models \phi$. Take now some $\omega \in \mathsf{2mod}(D_1(\Delta))$ s.t. $\widetilde{\ }$ is complete in ω. With Proposition 3, $\kappa^Z_\Delta(\omega{\downarrow}) = 0$ and thus $\omega{\downarrow} \models \phi$. With Definition 6, also $\omega \models \phi$. Thus, we have shown that for any $\omega \in \mathsf{2mod}(D_1(\Delta))$ s.t. $\widetilde{\ }$ is complete in ω, $\omega \models \phi$ which implies $D_1(\Delta) \mathrel{|\!\sim}^{\cap,c}_{\mathsf{2mod}} \phi$.

Suppose now that $D_1(\Delta) \mathrel{|\!\sim}^{\cap,c}_{\mathsf{2mod}} \phi$, i.e. for every $\omega \in \mathsf{2mod}(D_1(\Delta))$ s.t. $\widetilde{\ }$ is complete in ω, $\omega' \models \phi$. Take now some $\omega \in \Omega(\mathsf{At})$ s.t. $\kappa^Z_\Delta(\omega) = 0$. With Lemma 2 $\omega{\uparrow} \in \mathsf{2mod}(D_1(\Delta))$ and with Fact 1, $\widetilde{\ }$ is complete in $\omega{\uparrow}$. Thus, $\omega{\uparrow} \models \phi$. With Lemma 3, this implies that $\omega \models \phi$. Thus we have shown that for every $\omega \in \Omega(\mathsf{At})$, $\kappa^Z_\Delta(\omega) = 0$ implies $\omega \models \phi$, which implies that $\Delta \mathrel{|\!\sim}_Z \phi$. $\square$

3.5 Other Semantics

In this section we show that other semantics also respect Δ. We first investigate the two-valued stable semantics and then move to the three-valued complete, preferred and grounded semantics.

Stable Semantics We first notice that not every two-valued model of $D_1(\Delta)$ is stable:

Example 7. *Let $\Delta = \{(p|q), (q|p)\}$. Then $D_1(\Delta) = (\{p, q, \widetilde{p}, \widetilde{q}\}, L, C)$ with $C_p = \neg\widetilde{p} \wedge q$, $C_q = \neg\widetilde{q} \wedge p$ and $C_{\widetilde{x}} = \neg x$ for any $x \in \{p, q\}$.*

Notice that ω with $\omega(p) = \omega(q) = \top$ and $\omega(\widetilde{p}) = \omega(\widetilde{q}) = \bot$ is a two-valued model of $D_1(\Delta)$. It is, however, not stable. To see this, notice that $(D_1(\Delta))^\omega = (\{p, q\}, L, C^\omega)$ with $C^\omega_p = \top \wedge q$ and $C^\omega_q = \top \wedge p$. The grounded extension v of $(D_1(\Delta))^\omega$ assigns $v(p) = v(q) = u$.

Furthermore, stable models might be incomplete w.r.t. $\widetilde{\ }$, just like the two-valued models:

Example 8. *Recall the conditional knowledge base from Example 6. There, $v_3 \in \mathsf{2mod}(D_1(\Delta))$ with $v_3(p) = \top$ and $v_3(\widetilde{p}) = v_3(q) = v_3(\widetilde{q}) = \bot$. We have $(D_1(\Delta)^{v_3}) = (\{p\}, L, C^{v_3})$ with $C_p = \neg\bot$. Since the grounded extension v of $(D_1(\Delta)^{v_3}) = (\{p\}, L, C^{v_3})$ assigns $v(p) = \top$, we see that v_3 is stable. As was argued in Example 6, $\widetilde{\ }$ is incomplete in v_3.*

However, we can make some immediate observations about the stable models of $D_1(\Delta)$. We first recall the following result:

Theorem 2 ((Brewka et al. 2017, Theorem 3.1))**.** *For any* ADF *D,* $\mathsf{stable}(D) \subseteq \mathsf{2mod}(D)$.

It follows from Theorem 2 and Proposition 2 that every stable model of $D_1(\Delta)$ for which $\widetilde{\ }$ is complete, respects Δ:

Proposition 4. *Let a literal conditional knowledge base Δ and some $(\phi|\psi)$ be given. Then for any $\omega \in \mathsf{stable}(D_1(\Delta))$, if $\omega \models \ulcorner\psi\urcorner$ then $\omega \models \ulcorner\phi\urcorner$.*

We can furthermore show that any stable model of $D_1(\Delta)$ is maximally plausible according to κ^Z_Δ (modulo the $\downarrow$-transformation):

Proposition 5. *Let a literal conditional knowledge base Δ and an $\omega \in \mathsf{stable}(D_1(\Delta))$ for which $\widetilde{\ }$ is complete be given. Then $\kappa^Z_\Delta(\omega\downarrow) = 0$.*

Proof. Follows from Theorem 2 and Proposition 7. □

Three-Valued Semantics For all of the well-known three-valued semantics, we can show (just like for the two-valued and stable models) that any corresponding interpretation of the translation $D_1(\Delta)$ respects Δ (thus generalizing Proposition 2):

Proposition 6. *Let a literal conditional knowledge base Δ and a model $v \in \mathcal{V}$ of $D_1(\Delta)$ be given. Then for any $(\phi|\psi) \in \Delta$, if $v(\ulcorner\psi\urcorner) = \top$ then $v(\ulcorner\phi\urcorner) = \top$.*

Proof. Suppose that $v \in \mathcal{V}$ is a model and let $(\phi|\psi) \in \Delta$. Suppose that $v(\ulcorner\psi\urcorner) = \top$. Since v is a model, $v(\ulcorner\psi\urcorner) = \top$ implies $v(C_{\ulcorner\psi\urcorner}) = \top$. Since $(\phi|\psi) \in \Delta$, $C_{\ulcorner\psi\urcorner} = \neg - \ulcorner\psi\urcorner \wedge \ulcorner\phi\urcorner \wedge \bigwedge_{(\phi'|\psi)\in\Delta\setminus\{(\phi|\psi)\}} \ulcorner\phi'\urcorner$, and thus $v(C_{\ulcorner\psi\urcorner}) = \top$ implies $v(\ulcorner\phi\urcorner) = \top$. □

Corollary 2. *Let a literal conditional knowledge base Δ and some $(\phi|\psi) \in \Delta$ be given. Then:*

1. *For any* $\mathsf{sem} \in \{\mathsf{complete}, \mathsf{preferred}\}$ *and $v \in \mathsf{Sem}(D_1(\Delta))$, v respects Δ.*
2. *$v_g^{D_1(\Delta)}$ respects Δ.*[2]

3.6 Extended Literal Conditional Knowledge Bases

Since in our translation D_1, a conditional $(\phi|\psi)$ results in a support link from ϕ to ψ, it is not immediately clear how to translate a normality statement of the form $(\phi|\top)$, among others since $\top$ will not correspond to a node in the ADF. We circumvent this problem by modelling normality statements $(\phi|\top)$ by requiring that $-\ulcorner\phi\urcorner$ is not believed, i. e. by setting $C_{-\ulcorner\phi\urcorner} = \bot$. This results in the following translation for extended literal conditional knowledge bases:

Definition 7. *Given an extended literal conditional knowledge base Δ, we define: $D_1^{\mathsf{elcb}}(\Delta) = (\mathsf{Lit}\text{\textasciitilde}(\mathsf{At}(\Delta)), L, C)$ where: for any $\phi \in \mathsf{Lit}\text{\textasciitilde}(\mathsf{At}(\Delta))$,*

$$C_\phi = \begin{cases} \bot & \textit{if } \exists(\llcorner -\phi \lrcorner|\top) \in \Delta \\ \neg - \phi \wedge \bigwedge_{(\psi|\llcorner\phi\lrcorner)\in\Delta} \ulcorner\psi\urcorner & \textit{otherwise} \end{cases}$$

We notice that the first case can be expanded into the following form (where $\phi \in \mathsf{At}$):

- $C_\phi = \bot$ if there is some $(\neg\phi|\top) \in \Delta$

[2]Recall that $v_g^{D_1(\Delta)}$ denotes the grounded extension of $D_1(\Delta)$.

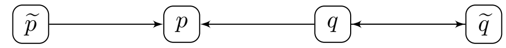

Figure 3: Graph representing the links between nodes of $D_1^{\mathsf{elcb}}(\Delta)$ in Example 9.

- $C_{\widetilde{\phi}} = \bot$ if there is some $(\phi|\top) \in \Delta$

We illustrate $D_1^{\mathsf{elcb}}(\Delta)$ with an example:

Example 9. *Let $\Delta = \{(p|\top), (q|p)\}$. Then $D_1^{\mathsf{elcb}}(\Delta) = (\{p, \widetilde{p}, q, \widetilde{q}\}, L, C)$ with $C_p = \neg\widetilde{p} \wedge q$, $C_{\widetilde{p}} = \bot$ and $C_x = \neg - x$ for any $x \in \{q, \widetilde{q}\}$. We have two two-valued models, v_1 and v_2 with: $v_1(p) = v_1(q) = \top$, $v_1(\widetilde{p}) = v_1(\widetilde{q}) = \bot$, $v_2(\widetilde{q}) = \top$ and $v_2(p) = v_2(q) = v_2(\widetilde{p}) = \bot$. Even though this option gives rise to an incomplete interpretation, v_2, there is no two-valued interpretation of $D_1^2(\Delta)$ that falsifies any rule in Δ. This is no coincidence as we show below.*

We now show the adequacy of D_1^{elcb} for extended literal knowledge bases:

Proposition 7. *Given an extended literal conditional knowledge base Δ and an $\omega \in \mathsf{2mod}(D_1^{\mathsf{elcb}})$ for which $\widetilde{\ }$ is complete in ω be given. Then $\kappa^Z_\Delta(\omega\downarrow) = 0$.*

Proof. Suppose Δ is an extendend literal conditional knowledge base and $\widetilde{\ }$ is complete in ω. We show that $\omega\downarrow \not\models \psi\wedge\neg\phi$ for any $(\phi|\psi) \in \Delta$, which with Lemma 1 implies the Proposition. We show the claim for $\psi = \top$, since the case where $\psi \neq \top$ is identical to the proof of Proposition 3. Thus consider $(\phi|\top) \in \Delta$. Since this means with Definition 7, $C_{-\ulcorner\phi\urcorner} = \bot$ and $\widetilde{\ }$ is complete in ω, $\omega \models \phi$. With Definition 6, this means $\omega\downarrow \models \phi$. □

Proposition 8. *Given an extended literal conditional knowledge base Δ and an $\omega \in \Omega(\mathsf{At})$, if $\kappa^Z_\Delta(\omega) = 0$ then $\omega\uparrow \in \mathsf{2Mod}(D_1^{\mathsf{elcb}})$.*

Proof sketch. Suppose that $\phi \in \{\psi, \widetilde{\psi} \mid \psi \in \mathsf{At}\}$ and there is some $(\llcorner -\phi \lrcorner|\top) \in \Delta$ (and thus $C_\phi = \bot$) and $\omega\uparrow \models \phi$. Since $\kappa^Z_\Delta(\omega) = 0$, $(\llcorner -\phi \lrcorner|\top) \in \Delta$ implies that $\omega \models \llcorner -\phi \lrcorner$, which with Definition 6 implies $\omega\uparrow \models -\phi$, contradicting $\omega\uparrow \models \phi$ and Proposition 1. Thus, for any $\phi \in \{\psi, \widetilde{\psi} \mid \psi \in \mathsf{At}\}$ for which there is some $(\llcorner -\phi \lrcorner|\top) \in \Delta$: $\omega\uparrow \models \phi$ iff $\omega\uparrow \models C_\phi$. The other case is identical to the proof of Lemma 2. □

The proof of the following Theorem, stating the inferential equivalence of D_1^{elcb} w.r.t. $\mid\!\sim^{\cap,c}_{\mathsf{2mod}}$ is completely analogous to the proof of Theorem 1:

Theorem 3. *Given an extended literal conditional knowledge base Δ, $\Delta \mid\!\sim_Z \phi$ iff $D_1^{\mathsf{elcb}}(\Delta) \mid\!\sim^{\cap,c}_{\mathsf{2mod}} \phi$.*

The reader might wonder why we did not simply set $C_\phi = \top$ for any $(\phi|\top) \in \Delta$. This would result in an inadequate translation, since any information about conditionals with ϕ as an antecedent would be removed from the ADF, as illustrated by the following example.

Example 10 (Example 9 continued). *We consider* $\Delta = \{(p|\top), (q|p)\}$ *(as in Example 9). If we translated this knowledge base using* D_1 *and by in addition setting* $C_p = \top$ *from above, we get:* $D'(\Delta) = (\{p, \widetilde{p}, q, \widetilde{q}\}, L, C)$ *with* $C_p = \top$ *and* $C_x = \neg - x$ *for* $x \in \{\widetilde{p}, q, \widetilde{q}\}$. *In that case, there are two two-valued models,* v_3 *and* v_4 *with:* $v_3(p) = v_1(q) = \top$, $v_3(\widetilde{p}) = v_3(\widetilde{q}) = \bot$, $v_4(p) = v_4(\widetilde{q}) = \top$ *and* $v_4(\widetilde{p}) = v_4(q) = \bot$. *In that case, there is a (complete) two-value model, namely* v_2, *that validates p but not q, even though* $(q|p) \in \Delta$ *(in fact,* $(q|p)$ *is even in* Δ_0*).*

3.7 Properties of the Translation

(Gottlob 1994) proposed several desirable properties for translations between non-monotonic formalisms like *adequacy*, *polynomiality* and *modularity*. In Section 3.4 we already discussed *adequacy* in-depth and we have shown, that our translation is adequate on the level of beliefs for all semantics and for any extended literal knowledge base.

A translation satisfies *polynomiality* if the translation is calculable with reasonable bounds. It is easy to see, that our translation is *polynomial* in the length of the translated conditional knowledge base.

For *modularity* we follow the formulation of (Strass 2013) for a translation from *ADFs* to a target formalism, even though *modularity* was originally defined for translations between circumscription and default logic (Imielinski 1987). In other words *modular* means that "local" changes in the translated conditional knowledge base results in "local" changes in the translation. A minimal notion of modularity would be that if we have to syntactically disjoint conditional knowledge bases Δ_1 and Δ_2, then changes in Δ_1 will result only in changes to C_s for some $s \in \mathsf{Lit}^{\sim}(\mathsf{At}(\Delta_1))$. Clearly the translation presented in this paper is *modular*.

The biggest downside of this translation is the fact, that it is not *language-preserving* since we use a language extension in this translation to construct the ADFs.

Finally, it is clear, that this translation is *syntax-based*, in the sense that the translation $D_1(\Delta)$ can be derived purely on the basis of the logical form of the knowledge base Δ.

4 Design Choices

In this section we motivate some important design choices underlying our translation D_1, especially the extension of the language to include the negation $\widetilde{\ }$, the direction of supporting links resulting from conditionals $(\phi|\psi)$ in the translated conditional knowledge base and the restriction to literal conditional knowledge bases.

4.1 The necessity of $\widetilde{\ }$

The critical reader might wonder, given that ADFs allow for the negation $\neg$ to be used in formulating acceptance conditions for nodes, if a second negation $\widetilde{\ }$ is really needed? Indeed, a first proposal for a translation avoiding $\widetilde{\ }$ would be the following:

Definition 8. *Given a literal conditional knowledge base* Δ, *we let* $D_2(\Delta) = (\mathsf{At}(\Delta), L, C)$ *where:* $C_\phi = \bigwedge_{(\psi|\phi)\in\Delta} \psi$ *if there is some* $(\psi|\phi) \in \Delta$. *and* $C_\phi = \phi$ *otherwise.*

Such a translation would be inadequate since conditionals with negative antecedents are not taken into account. Thus, for example, $\overline{p}\,\overline{q} \in 2\mathsf{mod}(D_2(\{(p|\neg q)\})$ since $(p|\neg q)$ is not taken into account in C_q. We could propose making the following adjustment to avoid this:

Definition 9. *Given a literal conditional knowledge base* Δ, *we let* $D_3(\Delta) = (\mathsf{At}(\Delta), L, C)$ *where:* $C_\phi = \bigwedge_{(\psi|\phi)\in\Delta} \psi \wedge \bigwedge_{(\psi|\neg\phi)\in\Delta} \neg\psi$ *if there is some* $(\psi|\phi) \in \Delta$ *or some* $(\psi|\neg\phi) \in \Delta$ *and* $C_\phi = \phi$ *otherwise.*

However, since $2\mathsf{mod}(D_3(\{(q|p), (q|\neg p)\}) = \{\mathring{q}\overline{p}\}$, this also results in an inadequate translation, since $((q|\neg p))(\overline{qp}) = 0$ and thus $\kappa^Z_\Delta(\overline{qp}) = 1$. A third option would be to take:

Definition 10. *Given a literal conditional knowledge base* Δ, *we let* $D_4(\Delta) = (\mathsf{At}(\Delta), L, C)$ *where:* $C_\phi = \bigwedge_{(\psi|\phi)\in\Delta} \psi \vee \bigwedge_{(\psi|\neg\phi)\in\Delta} \neg\psi$ *if there is some* $(\psi|\phi) \in \Delta$ *or some* $(\psi|\neg\phi) \in \Delta$ *and* $C_\phi = \phi$ *otherwise.*

Notice that $2\mathsf{mod}(D_4(\{(q|p), (s|\neg p)\})$ contains $p\overline{q}s$. Since $((q|p))(p\overline{q}s) = 0$, this means D_4 is not an adequate translation. There are, of course, some other variations possible, which do, however, lead to similar inadequacies. We hope to have convinced the reader of the fact that any translation which is based purely on the syntax of conditional knowledge bases does require a second negation.[3]

4.2 Antecedents as Partial Sufficient Conditions

One guiding idea behind our translation D_1 is that, relative to a conditional knowledge base Δ, a node $\phi \in \mathsf{Lit}^{\sim}$ can be believed only if for every conditional $(\psi|\llcorner\phi\lrcorner) \in \Delta$, $\ulcorner\psi\urcorner$ is believed. In other words, the links go from the consequent $\ulcorner\psi\urcorner$ to the antecedent ϕ. One might wonder if adequacy is preserved when we let the links between nodes run from antecedent to consequent. Such an alternative translation could be the following:

Definition 11. *Given a literal conditional knowledge base* Δ, *we define:* $D_5(\Delta) = (\{\phi, \widetilde{\phi} \mid \phi \in \mathsf{At}(\Delta)\}, L, C)$ *where* $C_\phi = \neg\widetilde{\phi} \wedge \bigvee_{(\llcorner\phi\lrcorner|\psi)\in\Delta} \ulcorner\psi\urcorner$ *for any* $\phi \in \mathsf{Lit}^{\sim}$.

This translation is not adequate, however:

Example 11. *Let* $\Delta = \{(p|q), (\neg p|s)\}$. *Then* $D_5(\Delta) = (\{p, \widetilde{p}, q, \widetilde{q}, s, \widetilde{s}\}, L, C)$ *with:* $C_p = \neg\widetilde{p} \wedge q$, $C_{\widetilde{p}} = \neg p \wedge s$, $C_x = \neg - x$ *for any* $x \in \{q, \widetilde{q}, s, \widetilde{s}\}$. *We depicted the corresponding graph in Figure 4.*

Consider $v(q) = v(s) = v(\widetilde{p}) = \top$ *and* $v(\widetilde{q}) = v(\widetilde{s}) = v(p) = \bot$. *Then v is a two-valued model of* $D_3(\Delta)$ *(indeed, observe that* $v(C_p) = v(\neg\widetilde{p} \wedge q) = \bot$ *since* $v(\widetilde{p}) = \top$*). However, notice that* $\kappa^Z_\Delta(\overline{p}qs) = 1$ *since* $((p|q))(\overline{p}qs) = 0$. *Thus, two-valued models of* $D_5(\Delta)$ *might not correspond to*

[3] Since ADFs under two-valued model semantics are equi-expressive with propositional logic (Strass 2014), it is not hard to come up with a translation that is adequate. For example, it is straightforward to show the adequacy (under two-valued semantics) of the following translation. Let $D_\star(\Delta) = (\mathsf{Atoms}(\Delta), L, C)$ with:

$$C_\phi = \bigvee_{\kappa^Z_\Delta(\omega)=0 \text{ and } \omega\models\phi} \omega \wedge \bigwedge_{\kappa^Z_\Delta(\omega)>0 \text{ and } \omega\models\phi} \neg\omega \vee \bigvee_{\kappa^Z_\Delta(\omega)>0 \text{ and } \omega\models\neg\phi} \omega$$

for any $\phi \in \mathsf{At}(\Delta)$. But such a translation is dependent on the semantics of system Z and therefore is not syntax-based.

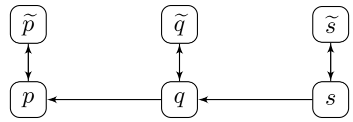

Figure 4: Graph representing the links between nodes of $D_5(\Delta)$ in Example 11.

maximally plausible worlds (even if the negation $\widetilde{\ }$ *is complete in such a model).*

4.3 Literal Conditionals

The final design choice made in this paper we motivate is the fact that we restricted attention to (possibly extended) literal conditional knowledge base as the object of translation. The reason is that we choose to represent conditionals $(\phi|\psi)$ as links between nodes ϕ and ψ (modulo transformation to the extend language). Moving to conditionals with arbitrary propositional formulas as antecedents and consequents would make it impossible to retain such a representation, since in abstract dialectical argumentation, nodes are essentially atomic.

5 Related Work

Our aim in this paper is to lay foundations of integrative techniques for argumentative and conditional reasoning. There are previous works, which have similar aims or are otherwise related to this endeavour. We will discuss those in the following.

First, there is huge body of work on *structured argumentation* (see e. g. (Besnard et al. 2014)). In these approaches, arguments are constructed on the basis of a knowledge base possibly consisting of conditionals. An attack relation between these arguments is constructed based on some syntactic criteria. Acceptable arguments are then identified by applying argumentation semantics to the resulting argumentation frameworks. Even though these formalisms also allow for argumentation-based inferences from a set of conditionals, these approaches will often give rise to inferences rather different from conditional logics. For example, in ASPIC$^+$ (Modgil and Prakken 2018), the knowledge base consisting solely of the defeasible rule $p \Rightarrow q$ will warrant no inference (in fact the set of arguments based on this knowledge base will be empty), whereas, for example, $D_1(\{(q|p)\}) \mathrel{|\!\sim}^{\cap,c}_{\mathsf{2mod}} \neg(p \wedge \neg q)$. This difference is caused by the fact that in structured argumentation, arguments are typically constructed in a proof-like manner. This means that defeasible rules can only be applied when there is positive evidence for the antecedent. Conditional logics, and our translation by extension, on the other hand, generate models that do not falsify any plausible conditional.

There have been some attempts to bridge the gap between specific structured argumentation formalisms and conditional reasoning. For example, in (Kern-Isberner and Simari 2011) conditional reasoning based on System Z (Goldszmidt and Pearl 1996) and DeLP (García and Simari 2004) are combined in a novel way. Roughly, the paper provides a novel semantics for DeLP by borrowing concepts from System Z that allows using *plausibility* as a criterion for comparing the strength of arguments and counterarguments. Our approach differs both in goal (we investigate the correspondence between argumentation and conditional logics instead of integrating insights from the latter into the former) and generality (DeLP is a specific and arguably rather peculiar argumentation formalism whereas ADFs are some of the most general formalism around).

Several works investigate postulates for nonmonotonic reasoning known from conditional logics (Kraus, Lehmann, and Magidor 1990) for specific structured argumentation formalisms, such as assumption-based argumentation (Čyras and Toni 2015; Heyninck and Straßer 2018) and ASPIC$^+$ (Li, Oren, and Parsons 2017). These works revealed gaps between nonmonotonic reasoning and argumentation which we try to bridge in this paper.

Besnard et al. (Besnard, Grégoire, and Raddaoui 2013) develop a structured argumentation approach where general conditional logic is used as the base knowledge representation formalism. Their framework is constructed in a similar fashion as the deductive argumentation approach (Besnard and Hunter 2008) but they also provide with *conditional contrariety* a new conflict relation for arguments, based on conditional logical terms. Even though insights from conditional logics are used in that paper, this approach stays well within the paradigm of structured argumentation.

In (Strass 2015) Strass presents a translation from an ASPIC-style defeasible logic theory to ADFs. While actually Strass embeds one argumentative formalism (the ASPIC-style theory) into another argumentative formalism (ADFs) and shows how the latter can simulate the former, the process of embedding is similar to our approach. However, inferentially the formalism of (Strass 2015) is more akin to ASPIC$^+$, in the sense that literals cannot be accepted unless there is some rule deriving them. Arguably, this formalism is more akin to D_5 (see Definition 4.2), as in the ADFs generated by (Strass 2015), rules result in support of the consequents of rules.

6 Outlook and Conclusion

In this paper we have presented and investigated a translation from conditional knowledge bases into abstract dialectical argumentation based on the syntatic similarities between the two frameworks. We provide an interpretation of plausible conditionals in abstract dialectical argumentation. We have shown that this interpretation is adequate under all of the well-known semantics for ADFs and have shown that the translation is polynomial and modular. Interestingly, the translation requires an extension of the language, which we have argued in Section 4 cannot be avoided.

Another limitation of our interpretation is that adequacy is only shown with respect to the level of beliefs $\mathit{Bel}\,(\kappa^Z_\Delta)$ (or equivalently the level of the most plausible worlds $(\kappa^Z_\Delta)^{-1}(0)$). In future work, we plan to investigate methods to obtain conditional inferences from ADFs and compare them with system Z. One proposal to do this is founded upon the *Ramsey-test* (Ramsey 2007), which says that a conditional $(\phi|\psi)$ is accepted if belief in ψ leads to belief in ϕ. Several ways of modelling the hypothetical belief in ψ are to be considered, such as revision by ψ (using e. g. revision of ADFs as proposed by (Linsbichler and

Woltran 2016)), observations of ϕ (Booth et al. 2012) or interventions with ϕ (Rienstra 2014). Furthermore, we plan to tackle the combination of the translation presented in this paper and the one from ADFs into conditional logics analyzed in previous works (Kern-Isberner and Thimm 2018; Heyninck, Kern-Isberner, and Thimm 2020). We want to answer the question what happens if we apply these translation one after each other. Finally, we plan to generalize the results of this paper to other conditional logics besides system Z, which we have chosen because of the many desirable properties it satisfies.

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