The semantical structure of conditionals, and its relation to formal argumentation

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Abstract

Conditionals, i. e. expressions of the logical form "if A, then B", have been a central topic of study ever since logic was on the academic menu. In contemporary logic, there is a consensus that the semantics of conditionals are best obtained by stipulating a subset of possible worlds in which the antecedent is true, and verifying whether the consequent is true in those worlds. Such a subset of possible worlds can represent, for example, the most typical worlds in which the antecedent is true. This idea has proven a fruitful basis, allowing for many systematic characterisation results as well as for making connections to other topics, such as belief revision and modal logic. In formal argumentation, the potential of these semantical ideas has not gone unnoticed in the last years, and many works have attempted to bridge the worlds of conditionals and arguments on the basis of these ideas. In this article, we give a thorough introduction to the semantics of conditionals and survey the adaptions of these semantics in the literature on computational argumentation, including structured argumentation and generalisations of abstract argumentation such as abstract dialectical frameworks. Furthermore, we highlight opportunities for future research on this topic.

1 Introduction

The study of conditionals has a long tradition in philosophy. This has resulted, among others, in a thorough and expansive formal study of conditionals, with a particular focus on semantical foundations of conditionals (see [54, 53] for an overview, and Section 3 for a summary). The basic idea underlying these semantics is that a conditional $(\psi|\phi)$ is accepted if ψ is true in a subset of ϕ-worlds. The details of this selection depend on the specific semantics. For example, non-monotonic conditionals of the form "if ϕ then typically ψ" define this selection in terms of the most plausible worlds [44].

The relation between argumentative formalisms and conditionals has been on the agenda of the computational argumentation community since its inception. On an intuitive level, an argument can be accepted only if it is sufficiently defended or supported, i. e., it seems that argumentation allows for a conditional interpretation. However, the investigation of connections between argumentation and conditionals on a more formal or semantical level provide a more nuanced perspective. It is well-known that argumentation and nonmonotonic resp. default logics are closely connected: In [19] it is shown that Reiter's default logic can be implemented by abstract argumentation frameworks, a most basic form of computational model of argumentation to which many existing approaches to formal argumentation refer. On the other hand, it is clear that argumentation allows for nonmonotonic, defeasible reasoning, and in [62] computational models of argumentation are assessed by formal properties that have been adapted from nonmonotonic logics. Furthermore, answer set programming [29] as one of the most successful nonmonotonic logics has often been used to implement argumentation [21, 15]. Nevertheless, argumentation and nonmonotonic reasoning are perceived as two different fields which do not subsume each other, and indeed, often attempts to transform reasoning systems from one side into systems of the other side have been revealing gaps that could not be closed (cf., e. g., [70, 42, 35]). While one might argue that this is due to the seemingly richer, dialectical structure of argumentation, in the end the evaluation of arguments often boils down to comparing arguments with their attackers, and comparing degrees of belief is a basic operation in qualitative nonmonotonic reasoning. In this chaper, we give a thorough introduction to the semantics of conditionals and provide an overview of work done on the comparison or incorporation of conditional semantics and argumentation.

Outline of this article In Section 2, we introduce the necessary preliminaries on propositional logic (Section 2.1), Kleene's Three-valued logic (Section 2.2) and abstract dialectical frameworks (Section 2.3). We provide an overview of the semantics of conditionals in Section 3, looking at semantics using selection functions (Section 3.1), systems of spheres (Section 3.2) and preferential models (3.3), while also pointing out connections with belief dynamics (Section 3.4). In Section 4, we look at work that investigates syntactic similarities between conditionals and argumentative formalisms. In Section 5, we overview approaches integrating conditional semantics in argumentative formalisms. In Section 5.3, we look at connections that have been made between structured accounts of argumentation and conditonal logics. In Section 6, we summarize further works that thematise conditionals and argumentation. A summary and outlook is provided in Section 7.

2 Preliminaries

In this section, we introduce the necessary preliminaries on propositional logic (Section 2.1), Kleene's Three-valued logic (Section 2.2) and abstract dialectical frameworks (Section 2.3).

2.1 Propositional Logic

For a (finite) set At of atoms let $\mathcal{L}(\mathsf{At})$ be the corresponding propositional language constructed using the usual connectives $\wedge$ (*and*), $\vee$ (*or*), $\neg$ (*negation*) and $\rightarrow$ (*material implication*). A (classical) *interpretation* (also called *possible world*) ω for a propositional language $\mathcal{L}(\mathsf{At})$ is a function $\omega : \mathsf{At} \rightarrow \{\mathsf{T}, \mathsf{F}\}$. Let $\Omega(\mathsf{At})$ denote the set of all interpretations for At. We simply write Ω if the set of atoms is implicitly given. An interpretation ω *satisfies* (or is a *model* of) an atom $a \in \mathsf{At}$, denoted by $\omega \models a$, if and only if $\omega(a) = \mathsf{T}$. The satisfaction relation $\models$ is extended to formulas as usual. As an abbreviation we sometimes identify an interpretation ω with its *complete conjunction*, i. e., if $a_1, \ldots, a_n \in \mathsf{At}$ are those atoms that are assigned T by ω and $a_{n+1}, \ldots, a_m \in \mathsf{At}$ are those propositions that are assigned F by ω we identify ω by $a_1 \ldots a_n \overline{a_{n+1}} \ldots \overline{a_m}$ (or any permutation of this). For example, the interpretation ω_1 on $\{a, b, c\}$ with $\omega(a) = \omega(c) = \mathsf{T}$ and $\omega(b) = \mathsf{F}$ is abbreviated by $a\overline{b}c$. For $\Phi \subseteq \mathcal{L}(\mathsf{At})$ we also define $\omega \models \Phi$ if and only if $\omega \models \phi$ for every $\phi \in \Phi$. We define the set of models $\mathsf{Mod}(X) = \{\omega \in \Omega(\mathsf{At}) \mid \omega \models X\}$ for every formula or set of formulas X. A formula or set of formulas X_1 *entails* another formula or set of formulas X_2, denoted by $X_1 \vdash X_2$, if $\mathsf{Mod}(X_1) \subseteq \mathsf{Mod}(X_2)$.

2.2 Kleene's Three-Valued Logic

Due to the three-valued nature of ADFs, we will need a three-valued logic to use as a basic logic underlying revision. Due to its high expressivity, we use Kleene's three-valued logic. A 3-valued interpretation for a set of atoms At is a function $v : \mathsf{At} \to \{\top, \bot, u\}$, which assigns to each atom in At either the value $\top$ (true, accepted), $\bot$ (false, rejected), or u (unknown). The set of all three-valued interpretations for a set of atoms At is denoted by $\mathcal{V}(\mathsf{At})$. We sometimes denote an interpretation $v \in \mathcal{V}(\{x_1, \ldots, x_n\})$ by $\dagger_1 \ldots \dagger_n$ with $v(x_i) = \dagger_i$ and $\dagger_i \in \{\top, \bot, u\}$, e.g., $\top\top$ denotes $v(a) = v(b) = \top$ for $\mathsf{At} = \{a, b\}$. A 3-valued interpretation v can be extended to arbitrary propositional formulas $\phi \in \mathcal{L}(\mathsf{At})$ via the truth tables in Table 1. We furthermore extend the language with a second, *weak negation* $\sim$, which is evaluated to true if the negated formula is false or undecided (i.e. there is no positive information for the negated formula). Thus, $\sim\phi$ means that no explicit information for ϕ being true ($v(\phi) \neq \top$) is given, whereas $\neg\phi$ means that ϕ is false ($v(\phi) = \bot$). The truth table for $\sim$ can also be found in Table 1.[1]

It will prove convenient to define the connective $\odot$ which stipulates a formula is undecided. We define $\odot\phi = \sim(\neg\phi \vee \phi)$. We define $\mathcal{L}^{\mathsf{K}}(\mathsf{At})$ as the language based on At, the unary connectives $\langle \neg, \sim, \odot \rangle$ and the binary connectives $\langle \wedge, \vee, \to \rangle$.

The following facts about $\sim$, which show some similarities between $\sim$ and classical negation, will prove useful below:

Fact 1. *For any $\phi \in \mathcal{L}^{\mathsf{K}}(\mathsf{At})$ and any $v \in \mathcal{V}(\mathsf{At})$: (1) $v(\sim\phi) \neq u$, and (2) $v(\sim\sim\phi) = \top$ iff $v(\phi) = \top$.*

We can show that $\odot$ expresses the undecidedness of any formula $\phi \in \mathcal{L}^{\mathsf{K}}$:

Fact 2. *For any $\phi \in \mathcal{L}^{\mathsf{K}}(\mathsf{At})$, $v(\odot\phi) = \top$ iff $v(\phi) = u$.*

We define the set of three-valued interpretations that satisfy a formula $\phi \in \mathcal{L}^{\mathsf{K}}(\mathsf{At})$ as $\mathcal{V}(\phi) = \{v \in \mathcal{V}(\mathsf{At}) \mid v(\phi) = \top\}$. A formula X_1 K-*entails* another formula X_2, denoted $X_1 \models_{\mathsf{K}} X_2$, if $\mathcal{V}(X_1) \subseteq \mathcal{V}(X_2)$. $X_1 \equiv_{\mathsf{K}} X_2$ iff $X_1 \models_{\mathsf{K}} X_2$ and $X_2 \models_{\mathsf{K}} X_1$.

[1] In the terminology of [73], the negation $\sim$ corresponds to Bochvar's *external negation* [11] and $\neg$ corresponds to Kleene's negation in his three-valued logic. $\sim$ is also referred to as Kleene's *weak negation* [75], since the conditions for $\sim\phi$ being satisfied are weaker than those for $\neg\phi$ being satisfied (i.e. $\{\neg\phi\} \models_{\mathsf{K}} \sim\phi$).

	$\neg$	$\sim$	$\odot$
$\top$	$\bot$	$\bot$	$\bot$
u	u	$\top$	$\top$
$\bot$	$\top$	$\top$	$\bot$

$\wedge$	$\top$	u	$\bot$
$\top$	$\top$	u	$\bot$
u	u	u	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

$\vee$	$\top$	u	$\bot$
$\top$	$\top$	$\top$	$\top$
u	$\top$	u	u
$\bot$	$\top$	u	$\bot$

Table 1: Truth tables for connectives in Kleene's K

Given an interpretation $v \in \mathcal{V}(\mathsf{At})$, we define:

$$\mathsf{form}(v) = \bigwedge_{v(a)=\top} a \wedge \bigwedge_{v(a)=\bot} \neg a \wedge \bigwedge_{v(a)=u} \odot a$$

Clearly, $\mathsf{form}(v)$ expresses exactly the beliefs expressed by a three-valued interpretation:

Fact 3. *For any $v \in \mathcal{V}(\mathsf{At})$ and any $a \in \mathsf{At}$: (1) $\mathsf{form}(v) \models_{\mathsf{K}} a$ iff $v(a) = \top$; (2) $\mathsf{form}(v) \models_{\mathsf{K}} \neg a$ iff $v(a) = \bot$; (3) $\mathsf{form}(v) \models_{\mathsf{K}} \odot a$ iff $v(a) = u$.*

2.3 Abstract Dialectical Frameworks

We briefly recall some technical details on ADFs following loosely the notation from [14]. An ADF D is a tuple $D = (\mathsf{At}, L, C)$ where At is a finite set of atoms, $L \subseteq \mathsf{At} \times \mathsf{At}$ is a set of links, and $C = \{C_s\}_{s \in \mathsf{At}}$ is a set of total functions (also called acceptance functions) $C_s : 2^{par_D(\mathsf{At})} \to \{\top, \bot\}$ for each $s \in \mathsf{At}$ with $par_D(s) = \{s' \in \mathsf{At} \mid (s', s) \in L\}$. An acceptance function C_s defines the cases when the statement s can be accepted (truth value $\top$), depending on the acceptance status of its parents in D. By abuse of notation, we will often identify an acceptance function C_s by its equivalent *acceptance condition* which models the acceptable cases as a propositional formula. In more detail, C_s expresses the conditions that are to be accepted for s to be accepted. $\mathfrak{D}(\mathsf{At})$ denotes the set of all ADFs $D = (\mathsf{At}, L, C)$.

Example 1. *We consider the following ADF $D_1 = (\{a, b, c\}, L, C)$ with $L = \{(a, b), (b, a), (a, c), (b, c)\}$ and $C_a = \neg b$, $C_b = \neg a$ and $C_c = \neg a \vee \neg b$. Informally, the acceptance conditions can be read as "a is accepted if b is not accepted", "b is accepted if a is not accepted" and "c is accepted if a is not accepted or b is not accepted".*

An ADF $D = (\mathsf{At}, L, C)$ is interpreted through 3-valued interpretations $\mathcal{V}(\mathsf{At})$ (see Section 2.2). Recall that $\Omega(\mathsf{At})$ consists of all the two-valued interpretations (i. e. interpretations such that for every $s \in \mathsf{At}$, $v(s) \in \{\top, \bot\}$). We define the information order $\leq_i$ over $\{\top, \bot, u\}$ by making u the minimal element: $u <_i \top$ and

$u <_i \bot$ and this order is lifted pointwise as follows (given two valuations v, w over At): $v \leq_i w$ iff $v(s) \leq_i w(s)$ for every $s \in \mathsf{At}$. The set of two-valued interpretations extending a valuation v is defined as $[v]^2 = \{w \in \Omega \mid v \leq_i w\}$. Given a set of valuations V, we denote with $\sqcap_i V$ the valuation defined by $\sqcap_i V(s) = v(s)$ if for every $v' \in V$, $v(s) = v'(s)$ and $\sqcap_i V(s) = u$ otherwise. $\Gamma_D : \mathcal{V}(\mathsf{At}) \mapsto \mathcal{V}(\mathsf{At})$ is defined as $\Gamma_D(v)(s) = \sqcap_i [v]^2(C_s)$. Intuitively, $\Gamma_D(v)$ assigns to an atom s the consensus of the truth values assigned by all completions of v to C_s.

For the definition of the stable model semantics, we need to define the reduct D^v of D given v, defined as: $D^v = (\mathsf{At}^v, L^v, C^v)$ with: (1) $L^v = L \cap (\mathsf{At}^v \times \mathsf{At}^v)$, and (2) $C^v = \{C_s[\{\phi \mid v(\phi) = \bot\}/\bot] \mid s \in \mathsf{At}^v\}$, where $C_s[\phi/\psi]$ is the formula obtained by substituting every occurrence of ϕ in C_s by ψ.

Definition 2.1. *Let $D = (\mathsf{At}, L, C)$ be an ADF with $v : \mathsf{At} \to \{\top, \bot, u\}$ an interpretation:*

- *v is a 2-valued model iff $v \in \Omega$ and $v(s) = v(C_s)$ for every $s \in \mathsf{At}$.*
- *v is* admissible for *D iff $v \leq_i \Gamma_D(v)$.*
- *v is* complete for *D iff $v = \Gamma_D(v)$.*
- *v is* preferred for *D iff v is $\leq_i$-maximally complete.*
- *v is* grounded for *D iff v is $\leq_i$-minimally complete.*
- *v is stable iff v is a model of D and $\{s \in \mathsf{At} \mid v(s) = \top\} = \{s \in \mathsf{At} \mid w(s) = \top\}$ where w is the grounded interpretation of D^v*[2]*.*

With $\mathsf{2val}(D)$, $\mathsf{admissible}(D)$, $\mathsf{complete}(D)$, $\mathsf{prf}(D)$, $\mathsf{grounded}(D)$, respectively $\mathsf{stable}(D)$ we denote the sets of two-valued, admissible, complete, preferred, grounded, respectively stable interpretations of D.

We finally define inference relations for ADFs:

Definition 2.2. *Given $\mathsf{Sem} \in \{\mathsf{prf}, \mathsf{grounded}, \mathsf{2val}, \mathsf{stable}\}$, an ADF $D = (\mathsf{At}, L, C)$ and $\phi \in \mathcal{L}^{\mathsf{K}}(\mathsf{At})$ we define: $D \mid\!\sim^{\cap}_{\mathsf{Sem}} \phi$ iff $v(\phi) = \top$ for all $v \in \mathsf{Sem}(D)$.*

Example 2 (Example 1 continued)**.** *The ADF of Example 1 has three complete models v_1, v_2, v_3 with:*

$$\begin{array}{lll} v_1(a) = \top & v_1(b) = \bot & v_1(c) = \top \\ v_2(a) = \bot & v_2(b) = \top & v_2(c) = \top \\ v_3(a) = u & v_3(b) = u & v_3(c) = u \end{array}$$

[2] [14] has show the grounded interpretation is uniquely defined for any ADF.

v_3 is the grounded interpretation whereas v_1 and v_2 are both preferred, two-valued and stable models.

Restricting ADFs to certain sub-classes based on the syntactic form of the acceptance conditions leads to representation of existing argumentative formalisms. One such formalism are the well known *Abstract argumentation frameworks* [19] where the only argumentative relation formalised is the one of attacks between arguments. In that case, acceptance conditions C_a are restricted to conjunctions of negations $\neg b_1 \wedge \ldots \wedge \neg b_n$, intuitively representing the attacks on the argument. For completeness, we include also the traditional definition of an argumentation framework:

Definition 2.3. *An* abstract argumentation framework *(AF) is a directed graph $AF = (A, R)$ where A is a finite set of* arguments *and R is an* attack relation *$R \subseteq A \times A$.*

For an AF $AF = (A, R)$, an argument a is said to *attack* an argument b if $(a, b) \in R$. We say that, an argument a is *defended by a set $E \subseteq A$* if every argument $b \in A$ that attacks a is attacked by some $c \in E$. For $a \in A$ we define

$$a^-_{AF} = \{b \mid (b, a) \in R\} \quad \text{and} \quad a^+_{AF} = \{b \mid (a, b) \in R\}.$$

In other words, a^-_{AF} is the set of attackers of a and a^+_{AF} is the set of arguments attacked by a. For a set of arguments $E \subseteq A$ we extend these definitions to E^+_{AF} and E^-_{AF} via $E^+_{AF} = \bigcup_{a \in E} a^+_{AF}$ and $E^-_{AF} = \bigcup_{a \in E} a^-_{AF}$, respectively. If the AF is clear in the context, we will omit the index.

An argumentation framework AF can be represented as the ADF $D_{AF} = (A, C)$ where $C_a = \bigwedge_{b \in a^-} \neg b$ for every $a \in A$. In that case, all of the traditional *extension-based semantics* [19] coincide with the ADF-semantics. It is interesting to notice furthermore that two-valued models and stable models coincide in the case of ADFs based on AFs. We will also call the sets of arguments labelled T according to a certain kind of labelling as the respective extension. For example, if the grounded labelling assigns T to the arguments a and c, we say that $\{a, c\}$ is the grounded extension.

We also mention here the notions of conflict-freeness and admissibility:

Definition 2.4. *Given $AF = (A, R)$, a set $E \subseteq A$ is*

- conflict-free *iff $\forall a, b \in E$, $(a, b) \notin R$;*
- admissible *iff it is conflict-free and it defends its elements.*

We use $cf(AF)$ and $ad(AF)$ for denoting the sets of conflict-free and admissible sets of an argumentation framework F, respectively.

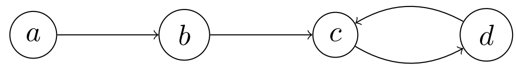


Figure 1: Abstract argumentation framework AF from Example 3.

Example 3. *Let* $AF = (\{a, b, c, d\}, \{(a, b), (b, c), (c, d), (d, c)\}$ *be an AF depicted as a directed graph in Figure 1. The sets* $\{a, c\}$ *and* $\{a, d\}$ *are the complete, preferred and stable extensions. While* $\{a\}$ *is the grounded extension.*

3 Semantics of conditionals

The study of the semantics of conditionals is concerned with statements of the form "if ϕ then ψ" as they are used in natural language. Several conditional logics have been developed with the aim of providing a semantics for conditional statements, and to study their properties. The aim of this section is to provide an introduction to the topic and an overview of the main approaches.

Of all the distinctions we can make among the types of conditionals that we use in everyday language, the most crucial distinction is that of *indicative* and *subjunctive* conditionals. While indicative conditionals make statements about what holds in the actual world, subjunctive conditionals make statements about hypothetical situations. The following example, due to [1], illustrates the difference.

1. If Oswald didn't kill Kennedy, then someone else did.

2. If Oswald hadn't killed Kennedy, then someone else would have.

Conditional (1) is an indicative conditional. It refers to the actual world where Oswald either did or did not kill Kennedy. It states that, in case Oswald did not kill Kennedy, someone else killed him. Conditional (2) is a subjunctive conditional. It presumes that Oswald did in fact kill Kennedy, and makes a claim about the hypothetical situation in which Oswald did not kill Kennedy. Subjunctive conditionals are also referred to as *counterfactuals.* Clearly, despite the similarities between (1) and (2), they make two very different claims. While it is quite reasonable believe that (1) is true, the truth of (2) is more contentious.

Indicative conditionals may, as a first approximation, be interpreted as material implications in propositional logic. According to this interpretation, the conditional 'if ϕ then ψ" ($\phi \rightarrow \psi$) is true unless ϕ is true and ψ is false. While this definition provides an adequate interpretation for conditionals as they are used in mathematical proofs, it is not satisfactory for indicative conditionals as they are used in natural language. This is due to a number of unintuitive consequences of the definition, such as that $\phi \to \psi$ is implied by ψ and by $\neg\phi$, that $\neg(\phi \to \psi)$ implies ϕ, and that for any ϕ and ψ, either $\phi \to \psi$ or $\psi \to \phi$ is true. Truth-functionality represents another issue. The material implication is truth-functional, since the truth value of $\phi \to \psi$ is a function of the truth values of ϕ and of ψ. Conditionals as they are used in natural language are not truth-functional. To see why, consider the sentences "John is happy" and "Mary is happy". Knowing the truth values of these two sentences does not imply that we know the truth value of the conditional "If Mary is happy then John is happy".

The study of the semantics of conditionals is driven by the need for more sophisticated accounts of the relationship between premises and conclusions of conditional statements. In the following subsections, we look at the two main semantical accounts of conditionals that have been developed since the sixties of the last century: semantics using selection functions (Section 3.1) and semantics using systems of spheres (Section 3.2). Thereafter, we look at the main semantical account of non-monotonic conditionals, namely the preferential models (3.3) and survey connections with belief dynamics (Section 3.4).

3.1 Selection Functions

The *selection function* approach, due to [68] and further developed by [49], represents one of the central ideas in the study of the semantics of conditionals. This approach is based on the idea that a conditional $\phi > \psi$ is true whenever ψ is true in the possible world where ϕ is true and which differs minimally from the actual world. The semantics is defined in terms of a *selection function* that represents a criterion to select such a possible world for any given antecedent ϕ and actual world w. Let L be a propositional language that, in addition to the usual propositional connectives, is closed under the conditional operator $>$. We will present the simplified formalisation of Stalnaker's semantics due to Nute [54]. A *Stalnaker model* is a quadruple $(I, R, s, [\cdot])$ where I is a set of possible worlds; $R \subseteq I \times I$ a binary reflexive accessibility relation; s a selection function; and $[\cdot]$ assigns to each sentence $\phi \in L$ a subset $[\phi]$ of I. The selection function s is a partial function that, if defined, assigns to a sentence ϕ and world $w \in I$ a world $s(\phi, w) \in I$. A selection function must satisfy the following conditions, which intuitively ensure that $s(\phi, w)$ can indeed be regarded as the world where ϕ is true that differ minimally from w.

1. $s(\phi, w) \in [\phi]$,
2. $(i, s(\phi, w)) \in R$,
3. If $s(\phi, w)$ is undefined then for all $w' \in I$ s.t. $(w, w') \in R$, $w' \notin [\phi]$,

4. If $w \in [\phi]$ then $s(\phi, w) = w$,
5. If $s(\phi, w) \in [\psi]$ and $s(\psi, w) \in [\phi]$, then $s(\phi, w) = s(\psi, w)$
6. $w \in [\phi > \psi]$ iff $s(\phi, w) \in [\psi]$ or $s(\phi, w)$ is undefined.

Example 4. *As an example, consider the model $I = \{bf, b\overline{f}, \overline{b}f, \overline{bf}\}$ with $R = I \times I$ and s partially defined by $s(b, b\overline{f}) = b\overline{f}$ and $s(b, w) = bf$ for every $w \in I \setminus \{b\overline{f}\}$. Then we see that $[b > f] = \{bf, \overline{b}f, \overline{bf}\}$ and thus $b > f$ is* not *true in $b\overline{f}$ but true in all other worlds.*

Given a Stalnaker model $(I, R, s, [\cdot])$, the conditional $\phi > \psi$ is true in world w whenever ψ is true in world $s(\phi, w)$ (more formally: $s(\phi, w) \in [\psi]$). The resulting logic, which Stalnaker refers to as **C2**, consists of those formulas that are true in every world of every model. The following set of properties provides an axiomatization of **C2**. More precisely, the logic **C2** coincides with the smallest set of formulas that is closed under the following two inference rules.

(RCEC) If $\phi \leftrightarrow \psi$ then $(\chi > \phi) \leftrightarrow (\chi > \psi)$.
(RCK) If $(\phi_1 \wedge \cdots \wedge \phi_n) \rightarrow \psi$ then
$((\chi > \phi_1) \wedge \cdots \wedge (\chi > \phi_n)) \rightarrow (\chi \rightarrow \psi)$ (for $n \geq 0$).

and contains all instances of the axioms:

(ID) $\phi > \phi$
(MP) $(\phi > \psi) \rightarrow (\phi \rightarrow \psi)$
(MOD) $(\neg\phi > \phi) \rightarrow (\psi > \phi)$
(CSO) $((\phi > \psi) \wedge (\psi > \phi)) \rightarrow ((\phi > \chi) \leftrightarrow (\psi > \chi))$
(CV) $((\phi > \psi) \wedge \neg(\phi > \neg\chi)) \rightarrow ((\phi \wedge \chi) > \psi)$
(CEM) $(\phi > \psi) \vee (\phi > \neg\psi)$

The main point of contention in Stalnaker's account concerns the CEM (Conditional Excluded Middle) axiom. This axiom states that, for every ϕ and ψ, either $\phi > \psi$ holds or $\phi > \neg\psi$ holds. Lewis offers the following counterexample to CEM [48]: Let A stand for "Bizet and Verdi are compatriots", F for "Bizet and Verdi are French", and I for "Bizet and Verdi are Italian". According to Lewis, we may well accept the conditional $A > F \vee I$ but reject both $A > F$ and $A > I$. If we accept $A > F \vee I$, however, then CEM forces us to accept either $A > F$ or $A > I$. CEM is closely related to what Nute refers to as the *Uniqueness Assumption* in Stalnaker's semantics [54]: for every antecedent ϕ and world w, there is exactly *one* world where ϕ is true and which differs minimally from w. Dropping CEM amounts to letting the selection function be a function that maps every antecedent ϕ and world w to a *set* of possible worlds. This option was pursued by [48]. He formalises a logic similar to Stalnaker's except that a conditional $\phi > \psi$ is taken to be true in world w if ψ is true in *all worlds* where ϕ is true and which differ minimally from w. The resulting logic, which Lewis calls **VC**, is axiomatised by the same set of inference rules and axioms as that of **C2** outlined above, except that CEM is replaced with CS (Conjunctive Sufficiency):

(CS) $\quad (\phi \wedge \psi) \rightarrow (\phi > \psi)$

While this logic drops the Uniqueness Assumption, it still relies on the questionable *Limit Assumption*: for every antecedent ϕ and world w, there is *at least one* world where ϕ is true and which differs minimally from w. Lewis' *system of spheres* semantics, described in the next section, does not rely on the limit assumption.

3.2 Systems of Spheres

Recall that, according to the selection function account, $\phi > \psi$ is true in world w whenever ψ is true in all worlds where ϕ is true and which differ minimally from w. Recall that the Limit Assumption requires that, for any antecedent ϕ and world w, there is at least one world where ϕ is true that differs minimally from w. Lewis points out that this assumption disagrees with situations where worlds get closer and closer to the actual world without end. This may happen if we consider antecedents such as "I am over 7 feet tall", where for any possible world where I am $7 + \epsilon$ feet tall, there is an even closer possible world where I am $7+\epsilon/2$ tall [48]. Lewis' *system of spheres* semantics provides an alternative semantics for conditionals that does not rely on the Limit Assumption, yet is characterised by the same axioms as the logic **VC** described above [48]. It is based on the idea that the conditional $\phi > \psi$ is true in world w whenever some world where both ϕ and ψ are true is closer then every world where ϕ and $\neg\psi$ are true. The formalisation of this idea requires a relative notion of closeness between worlds. A *sphere* around a world w is a set S that contains w and all worlds that are closer to w than every world not in S. A *system of spheres model* is a triple $(I, \$, [\cdot])$ where I and $[\cdot]$ are defined as before, and $\$$ maps each $w \in I$ to a nested set $\$_w$ of spheres around w. We can compare worlds according to their closeness to a world w as follows: if there is a sphere $S \in \$_w$ such that $w' \in S$ and $w'' \notin S$ then w' is more similar to w than w''. Given the model $(I, \$, [\cdot])$, the conditional $\phi > \psi$ is true in world w whenever either $\bigcup \$_w \cap [\phi]$ is empty, or there is an $S \in \$_w$ such that $S \cap [\phi]$ is not empty and $S \cap [\phi] \subseteq [\psi]$.

Example 5. *Consider the system of spheres (partially) defined by:* $\$_w = \{w\}, \{w, bf, \overline{b}\,\overline{f}, \overline{b}f\}, I$. *Then we see that in every world besides* $b\overline{f}$, $b > f$ *is true. In more detail, consider e.g. the world* $\overline{b}\,\overline{f}$. *As there is a sphere* $\{bf, \overline{bf}, \overline{b}f\} \in \$_{\overline{b}\,\overline{f}}$ *s.t.*

$\{bf, \overline{b}\,\overline{f}, \overline{b}f\} \cap [b] \subseteq [f]$. *More informally, the b-world closest to* $\overline{b}\,\overline{f}$ *is also an* f*-world, and thus, in* $\overline{b}\,\overline{f}$*, the conditional* $b > f$ *is true.*

3.3 Preferential Model Semantics

The preferential model semantics of [45] and [47] represents yet another approach to reasoning with conditionals. The main purpose of their approach, however, is to provide a semantics for non-monotonic consequence relations. A non-monotonic consequence relation is a relation $\mathrel{|\!\sim}$ between propositions having as its main characteristic that it violates, unlike the classical $\vdash$, the Monotony property:

(Monotony) If $\phi \mathrel{|\!\sim} \psi$ then $\phi \wedge \chi \mathrel{|\!\sim} \psi$.

Monotony means that we never retract conclusions when further information becomes known. However, in common sense reasoning, we often do so. We may, for instance, conclude that birds fly (*bird* $\mathrel{|\!\sim}$ *flies*) but retract this conclusion if we learn that the bird in question is a penguin (*bird* $\wedge$ *penguin* $\mathrel{|\!\not\sim}$ *flies*). The preferential model approach represents one of the most influential approaches to the general problem of non-monotonic reasoning.

The connection between non-monotonic consequence relations and conditionals lies in the fact that we can regard a consequence relation $\mathrel{|\!\sim}$ as a "flat" (i.e., not allowing nested conditionals) conditional logic: $\phi > \psi$ if and only if $\phi \mathrel{|\!\sim} \psi$. Furthermore, as we will see, several properties considered in the preferential model approach correspond to properties that are discussed in the context of conditional logics. Monotony is the first example. [54] calls it *Strengthening Antecedents* and dismisses it as invalid for any logic of subjunctive conditionals.

We will now provide an overview of the approach. We start with the model theory, which provides a semantics for non-monotonic inference relations. These models consist of a preference relation $\prec$ over states, where each state is labelled with a set of possible worlds. The preference relation can be thought of as an agent's belief about the relative degree of normality of states: if $s \prec s'$ then state s is more normal than state s'. The agent is willing to conclude ψ from ϕ if all most preferred states that satisfy ϕ also satisfy ψ. There are four classes of such models, each putting additional restrictions on the preference relation or state labelling. *Cumulative models* form the most general class:

Definition 3.1. *A* cumulative model over a set V of valuations *is a triple* $W = (S, \prec, l)$*, where* S *is a set containing elements called* states, $\prec$ *is a binary relation over* S*,* l *is a function mapping every state* $s \in S$ *to a non-empty set* $l(s) \subseteq V$*,* $(S, \prec, l)$ *satisfies the* smoothness *condition defined below. For every formula* $\phi \in \mathtt{lang}$ *we define* $\widehat{\phi}$ *by* $\widehat{\phi} = \{s \in S \mid \forall v \in l(s), v \models \phi\}$*. A state* s *is said to be* $\prec$-minimal *in a* *set $X \subseteq S$ iff $s \in X$ and there is no $s' \in X$ such that $s' \prec s$. Furthermore, W is called* finite *iff S is finite.*

The smoothness condition is related to the Limit Assumption discussed in Section 3.1. It ensures that, for every formula ϕ, it is possible to determine the preferred states in ϕ.

Definition 3.2. *A triple $(S, \prec, l)$ satisfies the* smoothness condition *iff for all $\phi \in$* `lang` *and $s \in \widehat{\phi}$, either s is $\prec$-minimal in $\widehat{\phi}$, or there is some $s' \in \widehat{\phi}$ such that s' is $\prec$-minimal in $\widehat{\phi}$ and $s' \prec s$.*

The following definition defines three restricted classes of cumulative models. Ordered and preferential models were defined by [45]. Ranked models were defined by [47].

Definition 3.3. *A cumulative model $W = (S, \prec, l)$ is:*

- ordered *if $\prec$ is a strict partial order.*
- preferential *if it is ordered and for all $s \in S$, $l(s)$ is a singleton.*
- ranked *if it is preferential and there exists a mapping $R : S \to \mathbb{N}$ such that $s \prec s'$ iff $R(s) < R(s')$.*

Definition 3.4. *A triple $W = (S, \prec, l)$ determines a consequence relation (denoted by $\mid\!\sim_W$) by the following rule:*

$$\phi \mid\!\sim_W \psi \textit{ iff for all } s \ \prec\textit{-minimal in } \widehat{\phi} \textit{ we have } \forall v \in l(s), v \models \psi.$$

We now move on to the axiomatisation of the four classes of models just defined. Consider the following set of properties.

(Reflexivity)	$\phi \mid\!\sim \phi$
(Left Logical Equivalence)	If $\phi \equiv \psi$ and $\phi \mid\!\sim \chi$ then $\psi \mid\!\sim \chi$
(Right Weakening)	If $\phi \mid\!\sim \psi$ and $\psi \models \chi$ then $\phi \mid\!\sim \chi$
(Cut)	If $\phi \mid\!\sim \psi$ and $\phi \wedge \psi \mid\!\sim \chi$ then $\phi \mid\!\sim \chi$
(Cautious Monotony)	If $\phi \mid\!\sim \psi$ and $\phi \mid\!\sim \chi$ then $\phi \wedge \psi \mid\!\sim \chi$
(Loop)	If $\phi_0 \mid\!\sim \phi_1$, $\phi_1 \mid\!\sim \phi_2$, ..., $\phi_{k-1} \mid\!\sim \phi_k$, $\phi_k \mid\!\sim \phi_0$ then $\phi_0 \mid\!\sim \phi_k$
(Or)	If $\phi \mid\!\sim \chi$ and $\psi \mid\!\sim \chi$ then $\phi \vee \psi \mid\!\sim \chi$
(Rational Monotony)	If $\phi \not\mid\!\sim \neg\psi$ and $\phi \mid\!\sim \chi$ then $\phi \wedge \psi \mid\!\sim \chi$

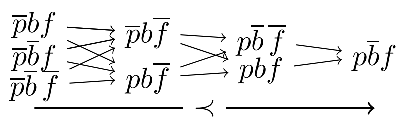
Let us point out that Reflexivity corresponds to the ID axiom of C2 and that Rational Monotony corresponds to the CV axiom. The set of axioms Reflexivity, Right

Weakening, Left Logical Equivalence, Cut, Cautious Monotony, and Or have become known as *system P* [45] and is considered as kind of a gold standard for nonmonotonic inference relations. The correspondence between these axioms and the four classes of models is established by the following Theorem 3.6. The axiomatisation of cumulative, ordered and preferential models is due to [45]. The axiomatisation of ranked models is due to [47].

Definition 3.5. *A consequence relation $\mid\!\sim$ is said to be:*

- cumulative *iff it satisfies Reflexivity, Right Weakening, Left Logical Equivalence, Cut and Cautious Monotony.*
- loop-cumulative *iff it is cumulative and satisfies Loop.*
- preferential *iff it is loop-cumulative and satisfies Or.*
- rational *iff it is preferential and satisfies Rational Monotony.*

Example 6. *As an example of a cumulative model, consider S consisting of all possible worlds over the signature $\{p, b, f\}$ and $\prec$ ordered as follows:*



Then we see that e.g. $pb \mid\!\sim_W \overline{f}$ as the verifying world pbf is $\prec$-preferred to the only falsifying world $pb\overline{f}$, i.e., $pbf \prec pb\overline{f}$.

An example of a rational model is given by the following order:

$$\overline{p}bf,\quad \overline{p}\overline{b}f,\quad \overline{p}\overline{b}\,\overline{f} \quad\prec\quad pb\overline{f},\quad \overline{p}b\overline{f} \quad\prec\quad pbf,\quad p\overline{b}\,\overline{f},\quad p\overline{b}f$$

We see here again that $pb \mid\!\sim_W \overline{f}$

Theorem 3.6. *Let $\mid\!\sim \subseteq \mathcal{L} \times \mathcal{L}$. It holds that $\mid\!\sim$ is cumulative (resp. loop-cumulative, preferential, rational) iff $\mid\!\sim$ is defined by a cumulative (resp. cumulative-ordered, preferential, ranked) model. Furthermore, if $\mathcal{L}$ is logically finite (i.e., contains a finite number of atoms) and $\mid\!\sim$ is cumulative (resp. loop-cumulative, preferential, rational) then $\mid\!\sim$ is defined by a* finite *cumulative (resp. cumulative-ordered, preferential, ranked) model.*

We have seen that the KLM-framework offers a formal model of the semantics of defeasible conditionals. The framework, however, does not give an account of how to construct a cumulative model for a given conditional knowledge base (typically, many different cumulative, or even preferential or ranked models are possible). In more detail, given a set of conditionals Δ of the form $(\phi|\psi)$ (where $\phi, \psi \in \mathcal{L}$), we are interested in determining a unique cumulative model W s.t. for every $(\phi|\psi) \in \Delta$, $\phi \mid\!\sim_W \psi$, i.e. W accepts every conditional. Several approaches for constructing such a model, sometimes called *inductive inference operators*, have been studied in the literature [47, 32, 41]. Probably the best-known and most-studied (even though not necessarily the best-behaved) approach is known as *rational closure* [47] or system Z [32].

We focus on system Z defined as follows. A conditional $(\psi|\phi)$ is tolerated by a finite set of conditionals Δ if there is a possible world ω with $(\psi|\phi)(\omega) = 1$ and $(\psi'|\phi')(\omega) \neq 0$ for all $(\psi'|\phi') \in \Delta$, i. e. ω verifies $(\psi|\phi)$ and does not falsify any (other) conditional in Δ. The Z-partitioning $(\Delta_0, \dots, \Delta_n)$ of Δ is defined as:

- $\Delta_0 = \{\delta \in \Delta \mid \Delta \text{ tolerates } \delta\}$;
- $\Delta_1, \dots, \Delta_n$ is the Z-partitioning of $\Delta \setminus \Delta_0$.

For $\delta \in \Delta$ we define: $Z_\Delta(\delta) = i$ iff $\delta \in \Delta_i$ and $(\Delta_0, \dots, \Delta_n)$ is the Z-partioning of Δ. Finally, the ranking function κ^Z_Δ is defined via: $\kappa^Z_\Delta(\omega) = \max\{Z(\delta) \mid \delta(\omega) = 0, \delta \in \Delta\} + 1$, with $\max \emptyset = -1$. Notice that this ranking correspond to a cumulative model, which we denote by $W^Z(\Delta)$.

We now illustrate ranked models in general and system Z in particular with the well-known "Tweety the penguin"-example.

Example 7. *Let $\Delta = \{(f|b), (b|p), (\neg f|p)\}$. This conditional belief base has the following Z-partitioning: $\Delta_0 = \{(f|b)\}$ and $\Delta_1 = \{(b|p), (\neg f|p)\}$. This gives rise to the following κ^Z_Δ-ordering over the worlds based on the signature $\{b, f, p\}$:*

ω	κ^Z_Δ	ω	κ^Z_Δ	ω	κ^Z_Δ	ω	κ^Z_Δ
pbf	2	$pb\overline{f}$	1	$p\overline{b}f$	2	$p\overline{b}\,\overline{f}$	2
$\overline{p}bf$	0	$\overline{p}b\overline{f}$	1	$\overline{p}\overline{b}f$	0	$\overline{p}\overline{b}\,\overline{f}$	0

As an example of a (non-)inference, observe that e.g. $\top \mid\!\sim_{W^Z(\Delta)} \neg p$ and $p \wedge f \mid\!\not\sim_{W^Z(\Delta)} b$.

3.4 Belief Revision and the Ramsey Test

Another important area in knowledge representation is that of *belief change*, which is concerned with supplying a formal model of the change of a belief base. In the context of this article, this is particularly interesting as there exist strong relationships between belief change and conditional reasoning, as we will explain below.

3.4.1 Belief Revision

We now recall the AGM-approach to belief revision [2] as reformulated for propositional logic by [40]. The following postulates for revision operators $\star : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ are formulated:

(R1) $\phi \star \psi \vdash \psi$
(R2) If $\phi \wedge \psi$ is satisfiable, then $\phi \star \psi \equiv \psi \wedge \phi$
(R3) If ψ is satisfiable, then so is $\phi \star \psi$
(R4) If $\phi_1 \equiv \phi_2$ and $\psi_1 \equiv \psi_2$, $\phi_1 \star \psi_1 \equiv \phi_2 \star \psi_2$
(R5) $(\phi \star \psi) \wedge \mu \vdash \phi \star (\psi \wedge \mu)$
(R6) If $(\phi \star \psi) \wedge \mu$ is satisfiable, then $\phi \star (\psi \wedge \mu) \vdash (\phi \star \psi) \wedge \mu$

An important result is the semantical characterisation of such a belief revision operator. For such a characterisation, a function $f : \mathcal{L}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ that assigns to each propositional formula $\phi \in \mathcal{L}$ a total preorder $\preceq_\phi$ over $\Omega(\mathsf{At})$ is used. The revision of a formula ϕ by a formula ψ is then defined as the formula which has as models exactly the $\preceq_\phi$-minimal models that satisfy ψ.

Definition 3.7 ([40])**.** *Given a formula $\phi \in \mathcal{L}(\mathsf{At})$, a function $f : \mathcal{L}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ assigning preorders $\preceq_\phi$ over $\Omega(\mathsf{At})$ to every formula $\phi \in \mathcal{L}(\mathsf{At})$ is faithful iff:*

1. *For every $\phi \in \mathcal{L}(\mathsf{At})$, if $\omega, \omega' \in \mathsf{Mod}(\phi)$ then $\omega \not\prec_\phi \omega'$,*
2. *For every $\phi \in \mathcal{L}(\mathsf{At})$, if $\omega \in \mathsf{Mod}(\phi)$ and $\omega' \notin \mathsf{Mod}(\phi)$ then $\omega \prec_\phi \omega'$,*
3. *For every $\phi, \phi' \in \mathcal{L}(\mathsf{At})$, if $\phi \equiv \phi'$ then $\preceq_\phi = \preceq_{\phi'}$.*

In [40] the following representation theorem for an AGMrevision operator $\star$ was shown:

Theorem 3.8 ([40])**.** *An operator $\star : \mathcal{L}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathcal{L}(\mathsf{At})$ satisfies R1–R6 iff there exists a faithful mapping $f^\star : \mathcal{L}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ that maps each formula $\phi \in \mathcal{L}(\mathsf{At})$ to a total preorder s.t.:*

$$\mathsf{Mod}(\phi \star \psi) = \min_{f^\star(\phi)}(\mathsf{Mod}(\psi)) \tag{1}$$

3.4.2 The Ramsey Test

Close relationships between belief revision and conditional logics were noticed by means of the *Ramsey test* [61], which says that a conditional $(\psi|\phi)$ is valid if ψ is believed after revision with the antecedent ϕ. The Ramsey test also gave rise to impossibility results on the compatibility of belief revision and conditional reasoning [26]. However, when [40] showed that total preorders underlie AGM-belief revision in a fundamental and inevitable way, it was at once also established that belief revision, conditional logic, and nonmonotonic inference were shown to be fully compatible. They can thus be seen as two different sides of a single topic or mode of reasoning [27, 50], at least when restricted to propositional beliefs. Indeed, when moving to other kinds of belief revision (e. g. [33, 18]), weaker kinds of conditionals [34, 51] or other forms of nonmonotonic inference, these interrelations tend to break down or are not investigated. For example, for revision in Horn-theories, [18] has shown that rational revision operators cannot be straightforwardly represented in terms of total preorders, thus severing the link between belief revision and nonmonotonic inference. It was shown that for revision operators in Horn theories satisfying additional postulates, semantics in terms of total preorders are sound and complete, but no investigations in corresponding non-monotonic inference relations have been made.

4 Syntactic similarities between Conditionals and Argumentative Formalisms

In this section, we survey work that explores syntactic similarities between formalisms in formal argumentation and conditionals, such as [38, 39, 37]. We explain where similarities have been identified and point to relevant differences.

The reason for looking at the syntactic similarities between abstract dialectical frameworks and conditional logic is the following. Syntactically, both frameworks focus on pairs of objects such as (ϕ, ψ). In conditional logic, these pairs are interpreted as conditionals with the informal meaning "if ϕ is true then, usually, ψ is true as well" and written as $(\psi|\phi)$. In abstract dialectical frameworks, these pairs are interpreted as acceptance conditions, and interpreted as "if ϕ is accepted then ψ is accepted as well". The resemblance of these informal interpretations is striking, but both approaches use fundamentally different semantics to formalise these interpretations. In several papers [38, 39, 37] these syntactical similarities formed the basis of a comparison between abstract dialectical frameworks and conditional logics. In more detail, they asked the question of whether, and how we can interpret abstract dialectical frameworks in terms of conditional logic so that acceptance in the argumentative system is defined by a nonmonotonic inference relation based on conditionals. The main insights are that there is a gap between argumentation and conditional semantics when applying several intuitive translations, but that there exists a class of translations that preserve the semantics for the 2-valued model semantics of ADFs (and for other semantics under certain conditions on the ADFs).

Furthermore, none of the translations studied are adequate for the grounded semantics and for the preferred and stable semantics in general. In the rest of this section, we provide more details on these results.

The following summarizes the investigations by Heyninck et al on syntactic similarities between ADFs and conditionals [38]. Where S is a set of atoms and $\mathfrak{D}_S$ is the set of all ADFs defined on the basis of S (i.e. all ADFs $D = (S, L, C)$), and $(\mathcal{L}(S)|\mathcal{L}(S))$ is the set of all condtionals over the propositional language generated by S, we investigate mappings $\mathfrak{T} : \mathfrak{D}_S \to \wp((\mathcal{L}(S)|\mathcal{L}(S)))$ (for arbitrary S).

There is a whole family of translations from ADFs to conditional logics which are *prima facie* apt to express the links between nodes s and their acceptance conditions C_s:

- $\Theta_1(D) = \{(s|C_s) \mid s \in S\}$
- $\Theta_2(D) = \{(C_s|s) \mid s \in S\}$
- $\Theta_3(D) = \Theta_1(D) \cup \Theta_2(D)$
- $\Theta_4(D) = \Theta_1(D) \cup \{(\neg s|\neg C_s) \mid s \in S\}$
- $\Theta_5(D) = \{((C_s \equiv s)|\top) \mid s \in S\}$
- $\Theta_6(D) = \Theta_2(D) \cup \{(\neg C_s|\neg s) \mid s \in S\}$.
- $\Theta_7(D) = \{(\neg s|\neg C_s) \mid s \in S\} \cup \{(\neg C_s|\neg s) \mid s \in S\}$.

Notice that all of these translations are based on the idea that there is a strong connection between the acceptance of an acceptance condition C_s and the acceptance of the corresponding node s. Indeed, as [14] puts it: "each node s has an associated acceptance condition C_s specifying the exact conditions under which s is accepted". However, in this formulation, it is not specified (1) when a formula is true according to a three-valued interpretation (i.e. is $a \vee \neg a$ true according to an interpretation v with $v(a) = u$? Different three-valued logics give different answers to this question), (2) what to accept when there are conflicts between different acceptance conditions (e.g. if $C_a = \neg b$ and $C_b = \neg a$) and (3) under which conditions we are justified in rejecting a node. Therefore, we systematically investigate different forms of conditionals based on the common idea that "the influence a node may have on another node is entirely specified through the acceptance condition" [14].

We now explain in more detail every translation. Θ_1 formalizes the intuition that whenever the condition of a node s is believed, normally, s should be believed as well. Likewise, Θ_2 formalizes the idea that if a node is believed, its condition should be believed as well. Θ_3 combines the two aforementioned intuitions. Θ_4 is a slight variation on this idea, combining Θ_1 with the constraint that whenever the negation of a condition of a node is believed, the negation of the node itself should be believed as well. Θ_5 postulates that a node should be equivalent to its condition. Θ_6, formalizes the following intuition: if s is believed, C_s has to be believed, and if $\neg s$ is believed, $\neg C_s$ has to be believed as well. Finally, Θ_7 is a formalization of the idea that whenever the negation of a node, respectively the negation of the condition of a node is believed, the negation of the condition of the node, respectively the negation of the node should be believed. Note that Θ_1 has already been investigated to some small extent in [43]. These translations were investigated with respect to their adequacy in full detail in [38]. In more detail, the following notion of *adequacy* was used there:

Definition 4.1. *Let S be a set of atoms and $\mathfrak{T} : \mathfrak{D}_S \to \wp((\mathcal{L}(S)|\mathcal{L}(S)))$ be a translation from* ADFs *to conditional knowledge bases. We furthermore define $W \models \Delta$ iff $\phi \mathrel{|\!\!\sim}_W \psi$ for every $(\psi|\phi) \in \Delta$. $\mathfrak{T}$ is:*

- OCF-adequate with respect to Sem *if: for every $D = (S, L, C)$ there is some ranked model W s.t. (1) $W \models \mathfrak{T}(D)$ and (2) for every $s \in S$, $D \mathrel{|\!\!\sim}^{\cap}_{\mathsf{Sem}} s$ iff $\top \mathrel{|\!\!\sim}_W s$.*[3]

- Z-adequate with respect to Sem *if: for every $D = (S, L, C)$ and every $s \in S$ it holds that: $D \mathrel{|\!\!\sim}^{\cap}_{\mathsf{Sem}} s$ iff $\mathfrak{T}(D) \mathrel{|\!\!\sim}^{Z} s$.*

Intuitively, a translation is OCF-adequate if the beliefs sanctioned by some ranking that is a model of the translation correspond to the consequences of the translated ADF D under some semantics Sem. The general picture that emerges in [38] is that:

- the translations Θ_1 and Θ_2 are *not* OCF-adequate or Z-adequate under any ADF-semantics,

- the translations $\Theta_3, \ldots, \Theta_7$ are OCF-adequate and Z-adequate under the two-valued model semantics, and

- the translations $\Theta_3, \ldots, \Theta_7$ are *not* OCF-adequate and Z-adequate under any other ADF-semantics.

We refer to [38] for full formal details, but illustrate this here with a few simple examples.

[3]The term OCF-adequate comes from *ordinal conditional functions*, a particularly useful implementation of ranked models due to Spoh [66].

Example 8 (Z-Inadequacy of Θ_1 w.r.t. $\mathsf{2mod}$)**.** *We consider the following ADF $D_1 = (\{a,b,c\}, L, C)$. Notice that*

$$\Theta_1(D_1) = \{(b|\neg a), (a|\neg b), (c|\neg a \vee \neg b)\}$$

which has the following Z-ranking:

ω	κ^z_Δ	ω	κ^z_Δ	ω	κ^z_Δ	ω	κ^z_Δ
abc	0	$ab\overline{c}$	0	$a\overline{b}c$	0	$a\overline{bc}$	1
$\overline{a}bc$	0	$\overline{a}b\overline{c}$	1	$\overline{ab}c$	1	$\overline{abc}$	1

We therefore see that $\Theta_1(D_1) \not\mid\!\sim^Z c$ even though $D \mid\!\sim^\cap_{\mathsf{2mod}} c$ and thus Θ_1 is not Z-adequate with respect to the $\mathsf{2mod}$*-semantics.*

Example 9 (Z-Inadequacy of Θ_2 w.r.t. $\mathsf{2mod}$)**.** *We consider the following ADF $D_2 = (\{a,b,c\}, L, C)$ where:*

$$C_a = \neg b \qquad\qquad C_b = \neg a \qquad\qquad C_c = a \vee b$$

D_2 has three complete models v_1, v_2, v_3 with: $v_1(a) = v_2(b) = v_1(c) = v_2(c) = \top$, $v_1(b) = v_2(a) = \bot$ and $v_3(a) = v_3(b) = v_3(c) = u$. Only v_1 and v_2 are 2-valued.

Moving to $\Theta_2(D) = \{(\neg a|b), (\neg b|a), (a \vee b|c)\}$, we see that

$$(\kappa^Z_{\Theta_2(D)})^{-1}(0) = \{a\overline{b}c, a\overline{bc}, \overline{a}bc, \overline{a}b\overline{c}, \overline{abc}\}.$$

This means that $\Theta_2(D_2) \not\mid\!\sim^Z c$ even though $D \mid\!\sim^\cap_{\mathsf{2mod}} c$, i.e. Θ_2 is not Z-adequate with respect to the $\mathsf{2mod}$*-semantics.*

Example 10. *If we look at $D_2 = (\{a,b,c\}, L, C)$ from the previous example again, we see that*

$$\Theta_3(D_2) = \{(a|\neg b), (b|\neg a), (c|a \vee b), (\neg a|b), (\neg b|a), (a \vee b|c)\}.$$

We see that $(\kappa^Z_{\Theta_3(D)})^{-1}(0) = \{a\overline{b}c, \overline{a}bc\}$. This illustrates OCF-adequacy and Z-adequacy of $\Theta_3(D_2)$. For the other translations $\Theta_4, \ldots, \Theta_7$, a similar result holds.

This example also lets us illustrate the Z-inadequacy of these translations for the complete and grounded semantics. Indeed, as there is one complete model of D_2 v_3 with $v_3(a) = v_3(b) = v_3(c) = u$, we see that $D \not\mid\!\sim_{\mathsf{grounded}} a \vee b$ whereas $\Theta_2(D_2) \mid\!\sim^Z a \vee b$.

Likewise, for preferred semantics, all translations prove inadequate:

Example 11. *We consider the following ADF* $D_3 = (\{a,b,c\}, L, C)$ *where* $C_a = \neg b$, $C_b = \neg a$, *and* $C_c = \neg b \wedge \neg c$. *This ADF has the following unique 2-valued models:* $v(a) = v(c) = \bot$ *and* $v(b) = \top$. *If we consider e.g.* $\Theta_3(D_3) = \{(a|\neg b), (b|\neg a), (c|\neg b \wedge \neg c), (\neg b|a), (\neg a|b), (\neg b \wedge \neg c|c)\}$, *we see that* $\kappa^Z_{\Theta_3(D)})^{-1}(0) = \{\overline{a}b\overline{c}\}$ *which means* $\Theta_3(D_3) \mid\!\sim^Z b$. *However,* D_3 *has two preferred intepretations: one corresponds to the* $\kappa^Z_{\Theta_3(D)}$*-minimal world (*$v(a) = v(c) = \bot$ *and* $v(b) = \top$*), and a second preferred model is* v' *with* $v'(a) = \top$, $v'(b) = \bot$ *and* $v'(c) = u$. *Thus,* $D \mid\!\sim^\cap_{\mathsf{preferred}} b$.

5 Conditional semantics in argumentation

In this section, we survey work that applies ideas from conditional logic in formal argumentation.

5.1 Abstract argumentation

In the following, we discuss the works [64] and [65] which apply conditional logic semantics in abstract argumentation frameworks.

5.1.1 Non-Classical Semantics for Abstract Argumentation

Classical interpretations of propositional logic (and other classical logics) provide a simple interpretation for the elements in the signature of the logic: an interpretation (or possible world) ω either evaluates an atom $a \in \mathsf{At}$ to T or F. Similarly, classical interpretations of abstract argumentation frameworks (*extensions*) provide the same view on the acceptance status of arguments: either an argument a is contained in an extension E or it is not.

In conditional logics, interpretations provide more structure and are usually based on some form of rankings of classical interpretations wrt. their *plausibility*, such as with *ordinal conditional functions* (or *ranking functions*). In the following, we consider ranking functions for abstract argumentation, i. e., functions that assign a degree of plausibility to extensions. Such a ranking between sets of arguments allows us to reason in a more fine-grained manner than with extension-based semantics. Where in classical extension-based semantics, we can either say that a particular set of arguments is an extension or not, in the ranking-based approach, we can compare two sets (which are not necessarily extensions for a given semantics) on the basis of how close they are to being acceptable.

In order to approach the topic in a general manner, we first consider extension-ranking semantics by [64].

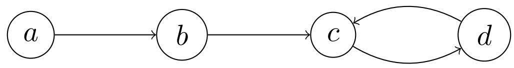


Figure 2: AF from Example 12.

Definition 5.1 (Extension-ranking semantics)**.** *Let $AF = (A, R)$ be an AF. An* extension ranking *on AF is a preorder*[4] $\sqsupseteq$ *over the power set of arguments 2^A. An* extension-ranking semantics τ *is a function that maps each AF to an extension ranking $\sqsupseteq^\tau_{AF}$ on AF.*

Note that extension rankings are not necessarily total. For an AF $AF = (A, R)$, an extension-ranking semantics τ, an extension ranking $\sqsupseteq^\tau_{AF}$, $E, E' \subseteq A$, and for $E \sqsupseteq^\tau_{AF} E'$ we say that E is *at least as plausible as* E' with respect to τ in AF. We introduce the usual abbreviations:

- E is *strictly more plausible than* E', denoted $E \sqsupset^\tau_{AF}$, if $E \sqsupseteq^\tau_{AF} E'$ but not $E' \sqsupseteq^\tau_{AF} E$;
- E and E' are *equally plausible*, denoted $E \equiv^\tau_{AF} E'$, if $E \sqsupseteq^\tau_{AF} E'$ and $E' \sqsupseteq^\tau_{AF} E$;
- E and E' are *incomparable*, denoted $E \asymp^\tau_{AF} E'$, if neither $E \sqsupseteq^\tau_{AF} E'$ nor $E \sqsupseteq^\tau_{AF} E'$.

To motivate the need of extension rankings further consider the following example.

Example 12. *Lets recall the AF from Example 3 with*

$$AF = (\{a, b, c, d\}, \{(a, b), (b, c), (c, d), (d, c)\}$$

and depicted in Figure 2. To compare the two sets $\{b\}$ and $\{c, d\}$ extension-based semantics such as admissible semantics do not provide a suitable solution. Both these sets are not admissible extensions, however $\{c, d\}$ is not even conflict-free, while $\{b\}$ is conflict-free. Therefore we argue that $\{b\}$ is a "better" set than $\{c, d\}$, since conflict-freeness is an undisputed property in the area of abstract argumentation. Extension-ranking semantics provides a suitable approach to rank $\{b\}$ and $\{c, d\}$.

Extension-based semantics provide a naive way of defining extension-ranking semantics. A set of arguments E is "better" than another set E' if the first set satisfies an extension-based semantics and the second set does not.

[4] A preorder is a (binary) relation that is *reflexive and transitive.*

Definition 5.2 (Least-discriminating extension-ranking semantics)**.** *Let $AF = (A, R)$ be an AF. Given an extension-based semantics σ, we define the* least-discriminating extension-ranking semantics *wrt. σ, denoted LD^σ by:*

- *$E \sqsupset_F^{LD^\sigma} E'$ if $E \in \sigma(F)$ and $E' \notin \sigma(F)$;*
- *and $E \equiv_F^{LD^\sigma} E'$, if $E, E' \in \sigma(AF)$ or $E, E' \notin \sigma(F)$.*

Example 13. *Continuing Example 12. Consider the two sets $\{a, c\}$ and $\{c, d\}$. $\{a, c\}$ is an admissible set, while $\{c, d\}$ is not even conflict-free, by using LD^{ad} we have $\{a, c\} \sqsupset_{AF}^{LD^{ad}} \{c, d\}$. So, the least-discriminating extension-ranking semantics is behaves in line with the binary classification of extension-based semantics. A set is either accepted or not wrt. an extension-based semantics σ i.e. a set is either part of the upper level if that set satisfies semantics σ or on the lower level if the set does not satisfy σ.*

5.1.2 Ordinal Conditional Functions for Abstract Argumentation

[65] introduced ranking functions for abstract argumentation frameworks. These ranking functions are a starting point for fully capturing the ideas and concepts of conditional logics in abstract argumentation. A ranking function $\kappa(I, O)$ for an $AF = (A, R)$ is used to compute a numerical plausibility value for a set of arguments $I \subseteq A$ to be considered *in* under the assumption that the set of arguments $O \subseteq A$ is considered *out*. Unlike ranking functions for conditional logics, ranking functions for argumentation frameworks need two parameters to compute a numerical plausibility value, since AFs do not have a notion of negation. E.g. in Example 12, the pair $(\{a, c\}, \{b, d\})$ can be seen as an analogue to the world $a\overline{b}c\overline{d}$.

Definition 5.3. *Let $AF = (A, R)$ be an AF. A* ranking function *for AF is a function $\kappa : 2^A \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$. For sets $I, O \subseteq A$ we abbreviate*

$$\kappa(I, O) = \min\{\kappa(S) | I \subseteq S, S \cap O = \emptyset\}$$
$$\kappa(I, O) = \infty \text{ if } I \cap O \neq \emptyset$$

Example 14. *Let $AF_2 = (\{a, b, c\}, \{(a, b), (b, c)\}$ be an AF and consider an exemplary ranking function κ. Since $\{a, c\}$ is the preferred extension we argue that $\{a, c\}$ should receive a plausibility value of 0, because no set is more plausible than a preferred extension, i.e. $\kappa(\{a, c\}, \emptyset) = \kappa(\{a, c\}, \{b\}) = 0$. Then the two admissible sets $\{a\}$ and $\emptyset$ should receive a plausibility value of 1, because these two sets are atleast admissible even-though they are not preferred, i.e. $\kappa(\{a\}, \{b, c\}) = \kappa(\{a\}, \{b\}) = \kappa(\{a\}, \{c\}) = \kappa(\{a\}, \emptyset) = 1$ and $\kappa(\emptyset, S) = 1$ for every*

$S \subseteq \{a, b, c\}$. *The two conflict-free sets* $\{b\}$ *and* $\{c\}$ *receive a plausibility value of* 2, *i.e.* $\kappa(\{b\}, \{a, c\}) = \kappa(\{b\}, \{a\}) = \kappa(\{b\}, \{c\}) = \kappa(\{b\}, \emptyset) = 2$ *and* $\kappa(\{c\}, \{a, b\}) = \kappa(\{c\}, \{a\}) = \kappa(\{c\}, \{b\}) = \kappa(\{c\}, \emptyset) = 2$. *Since the two sets* $\{a, b\}$ *and* $\{b, c\}$ *each entail only one conflict each, they should receive a better plausibility value than* $\{a, b, c\}$, *i.e.* $\kappa(\{a, b\}, \{c\}) = \kappa(\{a, b\}, \emptyset) = \kappa(\{b, c\}, \{a\}) = \kappa(\{b, c\}, \emptyset) = 3$ *and* $\kappa(\{a, b, c\}, \emptyset) = 4$.

Conditional logics semantics follows one single principle for conditional acceptance ("a conditional is accepted if its verification is more plausible than its violation"). On the other hand in abstract argumentation and in particular for admissible-based reasoning two guiding principles can be found:

Conflict-freeness: An argument should not be accepted if one of its attackers is accepted.

Reinstatement: An argument should be accepted if all its attackers are not accepted.

Conflict-freeness describes that a set should not contain two arguments that attack each other. So conflicting sets should be less plausible than conflict-free sets. *Reinstatement* describes that if there is no reason to reject an argument, then that argument should not be rejected. So a set which defends itself against all possible attackers is at least as plausible as a set that does not defend itself. The implementation of these two principles for ranking functions κ is:

Definition 5.4. *Let* $AF = (A, R)$ *be an AF,* $a, b \in A$, *and* κ *a ranking function.*

- κ accepts *an attack* (a, b) *with* $a \neq b$ *if* $\kappa(\{a\}, \{b\}) < \kappa(\{a, b\}, \emptyset)$.
- κ possibly reinstates *an argument* a *if* $\kappa(S \cup \{a\}, a^-) \leq \kappa(S, \{a\} \cup a^-)$ *for all* $S \subseteq A$ *with* $S \cap (a^- \cup a^+) = \emptyset$.

In other words, for an attack (a, b) to be accepted by a ranking function, it is more plausible for a to be *in* and b to be *out* than for both a and b to be *in* at the same time. An argument a is possibly reinstated by a ranking function if all attackers of a are *out*, then a being *in* should be at least as plausible as a being *out*.

If a ranking function satisfies the two principles for all arguments and all attacks of an AF, then that ranking function satisfies that AF.

Definition 5.5. *A ranking function* κ satisfies *an AF* $AF = (A, R)$ *if it accepts all attacks in* R *and possibly reinstates all arguments in* A.

i	$\kappa^{-1}(i)$
3	$(\{a,b\},\emptyset)$
2	$(\emptyset,\{a\})$, $(\emptyset,\{b\})$, $(\emptyset,\{a,b\})$
1	$(\{b\},\{a\})$, $(\{b\},\emptyset)$
0	$(\emptyset,\emptyset)$, $(\{a\},\emptyset)$, $(\{a\},\{b\})$

Table 2: Example ranking function for Example 15. Note κ is only partially defined.

Example 15. *Consider* $AF_3 = (\{a,b\},\{(a,b)\})$*. So the following statements have to hold for a ranking function* κ *to satisfy* AF_3*:*

1. $\kappa(\{a\},\{b\}) < \kappa(\{a,b\},\emptyset)$
2. $\kappa(\{a\},\emptyset) \leq \kappa(\emptyset,\{a\})$
3. $\kappa(\{b\},\{a\}) \leq \kappa(\emptyset,\{a,b\})$

Table 2 depicts a ranking function that satisfies AF_3*. The two admissible sets are* $\emptyset$ *and* $\{a\}$*, these two sets are also on the lowest level, meaning that these sets are the most plausible sets. If we compare the two not admissible sets* $\{b\}$ *and* $\{a,b\}$ *we see that* $\{b\}$ *is ranked higher than* $\{a,b\}$*. This behaviour is intuitive, since while both these sets are not admissible and* $\{b\}$ *is at least conflict-free.*

Note that if an AF contains any self-attacking argument a, then there can be no ranking function that satisfies that AF. This is because in order to accept the attack (a,a), it must hold that $\kappa(\{a\},\{a\}) < \kappa(\{a\},\emptyset)$, which is impossible since $\kappa(\{a\},\{a\}) = \infty$.

5.1.3 System Z Ranking function for Abstract Argumentation Frameworks

Next, we discuss a ranking function for AFs inspired by system Z. Recall that the basic idea of system Z is that a conditional $(\phi|\psi)$ is *tolerated* by a set of conditionals if it is confirmed by a possible world ω and no other conditional is refuted. When investigating an attack (a,b) in an argumentation framework, it can be concluded that if a is part of an extension E, then b should not be part of the same extension. Thus attacks between two arguments within an argumentation framework represent a conditional relation between those two arguments, i.e. for an attack (a,b) we can formulate: "if a is acceptable then b should not be acceptable". Therefore, we interpret the attack relation of an argumentation framework as a set of conditionals and to model the idea of system Z it has to hold that in order to *tolerate* an attack (conditional) we have to find a set of arguments (interpretation), which verifies that attack while not violating any other attack.

Definition 5.6. *Let $AF = (A, R)$ be an AF.*

- *A set $S \subseteq A$* verifies *an attack (a, b) iff $a \in S$ and $b \notin S$.*
- *A set $S \subseteq A$* violates *an attack (a, b) iff $a \in S$ and $b \in S$.*
- *A set $S \subseteq A$* satisfies *an attack (a, b) iff it does not violate it.*

Intuitively, a set of arguments satisfies an attack if this set does not contain both the attacker and the target of the attack. For AF $AF_2 = (\{a, b, c\}, \{(a, b), (b, c)\})$ from Example 14, we can observe that the set $S_1 = \{a\}$ verifies the attack (a, b) and does not violate the attack (b, c), while the set $S_2 = \{a, b\}$ verifies the attack (b, c), however S_2 violates attack (a, b).

Verifying an attack is not enough to capture the full picture of reasoning in abstract argumentation since only *conflict-freeness* is captured. To capture *reinstatement* as well the notion of defence has to be modelled with the toleration notion. For this purpose we add an another condition for an attack to be tolerated by a set of arguments, the so-called *attack admissibility.* To satisfy attack admissibility of an argument it has to hold that if all the attackers of the argument are *out*, then the argument itself should be *in.*

Definition 5.7. *Let $AF = (A, R)$ be an argumentation framework.*

- *A set $S \subseteq A$* verifies attack admissibility *of $a \in A$ iff $a \in S$ and $b \notin S$ for all $b \in a^-$.*
- *A set $S \subseteq A$* violates attack admissibility *of $a \in A$ iff $a \notin S$ and $b \notin S$ for all $b \in a^-$.*
- *A set $S \subseteq A$* satisfies attack admissibility *of $a \in A$ iff it does not violate it.*

Recall $AF_2 = (\{a, b, c\}, \{(a, b), (b, c)\})$, then set $S_3 = \{a, c\}$ verifies attack admissibility of argument c, because the only attacker of c, b is not part of S_3 and one of the attackers of b is contained in S_3. For set $\{b\}$ we have the case where argument a is not part of $\{b\}$, however $\{b\}$ also does not contain any attacker of a, hence attack admissibility of a is violated.

By combining these two definitions, we define when an attack can be tolerated.

Definition 5.8. *Let $AF = (A, R)$ be an argumentation framework. A set $P \subseteq R$* tolerates *an attack (a, b) iff there is a set $S \subseteq A$ that*

1. verifies (a, b)*,*

2. satisfies each attack in P*, and*

3. satisfies attack admissibility of each $c \in A$

To tolerate an attack, we have to find a set of arguments S that is conflict-free and every argument not in S has to be attacked. Recall $AF_2 = (\{a, b, c\}, \{(a, b), (b, c)\})$, then attack (b, c) is not tolerated by $\{(a, b), (b, c)\}$. For (b, c) to be verified for any set S, it must hold that $b \in S$. Then, to not violate (a, b) a is not allowed to be contained in S. However, then we have the problem that S does not contain any attackers of a, meaning that attack admissibility of a is violated.

With the help of the notion of toleration, we define a ranking function κ^Z inspired by system Z for AFs.

Definition 5.9. *Let* $AF = (A, R)$ *be an argumentation framework. Then the Z-attack-Partitioning* $(R_0, \dots, R_n)$ *with* $R_0 \cup \ldots \cup R_n \subseteq R$ *is defined as*

- $R_0 = \{r \in R \mid R \text{ tolerates } r\}$
- $(R_1, \dots, R_n)$ *is the Z-attack-Partitioning of* $R \setminus R_0$

For $r \in R$ *define* $Z_R(r) = i$ *if* $r \in R_i$ *and*

$$\kappa^Z(S, X) = \max\{Z(r) \mid S \text{ violates } r\} + 1$$

where $X \subseteq A$ *is any set s.t.* $S \cap X = \emptyset$*.*

Attacks in R_0 are tolerated with respect to the complete set of attacks R of an $AF = (A, R)$, while attacks in R_1 are tolerated only after removing the attacks in R_0. Using this partitioning of attacks we can rank the sets of arguments based on their plausibility with respect to the attacks. If a set violates an attack on level 0, while a different set violates an attack on level 1, then the first set is more plausible than the second one. The higher an attack is ranked, the worse its violation is. Thus, the partitioning of attacks can be interpreted as a split based on the impact of each attack in the AF, with attacks on lower ranks being considered better. It is therefore more important to satisfy a single high ranked attack than to satisfy several low ranked attacks.

Example 16. *Consider AF from Example 12. Then to tolerate attack* (b, c) *argument* b *has to be verified, however then attack admissibility of* a *is violated, hence*

i	$\kappa^{-1}(i)$
2	$(\{b,c\},X),(\{a,b,c\},X),(\{b,c,d\},X),(\{a,b,c,d\},X)$
1	$(\{a,b\},X),(\{c,d\},X),(\{a,b,d\},X),(\{a,c,d\},X)$
0	$(\emptyset,X),(\{a\},X),(\{b\},X),(\{c\},X),(\{d\},X),(\{a,c\},X),(\{b,d\},X),(\{a,d\},X)$

Table 3: κ^Z, where for every pair (I,X) $X \subseteq A$ is any set s.t. $I \cap X = \emptyset$.

$(b,c) \notin R_0$. *The remaining attacks are tolerated by* R*, so the Z-attack-Partitioning of* R *is* (R_0, R_1) *with*

$$R_0 = \{(a,b),(c,d),(d,c)\}$$
$$R_1 = \{(b,c)\}$$

Consider sets $\{a,c,d\}$ *and* $\{b,c\}$*, then* $\{a,c,d\}$ *violates* $(a,b),(c,d)$ *and* (d,c) *while* $\{b,c\}$ *violates* (b,c)*. Since* $(b,c) \in R_1$ *it holds that*

$$\kappa^Z_{AF}(\{a,c,d\},\emptyset) < \kappa^Z(\{b,c\},\emptyset).$$

Table 3 depicts κ^Z_{AF} *for AF from Example 12.*

5.1.4 Extension-ranking Semantics based on System Z

The ranking functions for AFs can be seen as a special instance of extension-ranking semantics. These functions allow us to rank sets of arguments based on their plausibility. So we can define an extension-ranking semantics based on the system Z ranking function for AFs by stating that a set of arguments E is at least as plausible as another set E' if κ^Z returns a lower value for E than for E' with respect to the remaining arguments.

Definition 5.10. *Let* $AF = (A,R)$ *be an AF and* $E, E' \subseteq A$*. Define the* system Z extension-ranking semantics $\sqsupseteq^{\kappa^Z}_{AF}$ *via*

$$E \sqsupseteq^{\kappa^Z}_{AF} E' \textit{ iff } \kappa^Z(E, A \setminus E) \leq \kappa^Z(E', A \setminus E').$$

In other words, E is at least as plausible as E', if E being *in*, while all other arguments not in E are considered *out* is more plausible than E' being considered *in* while all arguments not in E' are considered *out*.

Example 17. *Consider again AF from Example 12. Then Table 4 depicts the ranking corresponding to* $\sqsupseteq^{\kappa^Z}_{AF}$*. All conflict-free sets are part of the most plausible sets, while sets with conflicts are ranked lower. The number of conflicts is not as important as in the approaches of [64]. In their approaches,* $\{b,c\}$ *is always ranked strictly better than* $\{b,c,d\}$*. While for* κ^Z *these two sets are ranked equally.*

$$\emptyset \equiv^{\kappa^Z}_{AF} \{a\} \equiv^{\kappa^Z}_{AF} \{b\} \equiv^{\kappa^Z}_{AF} \{c\} \equiv^{\kappa^Z}_{AF} \{d\} \equiv^{\kappa^Z}_{AF} \{a,c\} \equiv^{\kappa^Z}_{AF} \{b,d\} \equiv^{\kappa^Z}_{AF} \{a,d\}$$
$$\sqsupset^{\kappa^Z}_{AF} \{a,b\} \equiv^{\kappa^Z}_{AF} \{c,d\} \equiv^{\kappa^Z}_{AF} \{a,b,d\} \equiv^{\kappa^Z}_{AF} \{a,c,d\}$$
$$\sqsupset^{\kappa^Z}_{AF} \{b,c\} \equiv^{\kappa^Z}_{AF} \{a,b,c\} \equiv^{\kappa^Z}_{AF} \{b,c,d\} \equiv^{\kappa^Z}_{AF} \{a,b,c,d\}$$

Table 4: Extension-ranking for AF based on $\sqsupseteq^{\kappa^Z}_{AF}$.

For further discussions about the system Z extension-ranking semantics we refer to [65].

In this subsection, we have seen that the ideas and concepts of ranking functions can be applied to abstract argumentation frameworks. Sets of arguments can be seen as interpretations and an attack between two arguments can be seen as a conditional i.e. for an attack (a, b) we say that if a is accepted, then b is not accepted. The results of this investigation are functions allowing us to compare sets of arguments based on their plausibility, which is in line with recent work on extension-ranking semantics [64].

5.2 Dynamic Conditionals for Abstract Dialectical Frameworks

Partially based on the differences between the semantics of ADFs and conditionals (Section 4), [36] defined conditional inference relations for ADFs. They took inspiration from the propositional setting, where there exist strong connections between *conditional inference* and *belief revision* as explained in Section 3.4.

For simplicity, we explain here the main ideas for two-valued ADF-semantics (in particular, the two-valued model and stable semantics). We refer to full details, and analogous results for three-valued semantics (e.g. complete, grounded and preferred) to [36].

5.2.1 Revising ADFs

Informally, [39] study the revision of argumentative contexts, which are represented by an ADF D, by new information, represented as logical formula ϕ, resulting in a revised argumentative context $D \star \phi$.

We concentrate on revising ADFs by formulas, resulting in a new ADF, i.e. revision operators $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}^{\mathsf{K}}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$. Revisions is always relative to a chosen semantics, and when this semantics is two-valued (e. g. two-valued models or stable models), we will restrict attention to revision by formulas in propositional logic in view of the two-valued nature of the mentioned semantics.

As an example of when this kind of revision can be useful, consider the following:

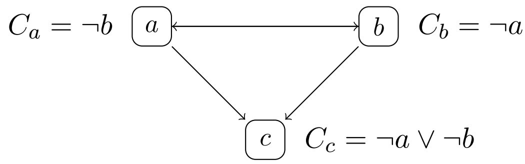


Figure 3: Argumentative representation of Example 18.

Example 18. *Consider making travel plans while being based in Germany. There are three candidate destinations: Addis Aba (Ethiopia), Boston (USA), and Cochem (Germany). There is not enough time to make two intercontinental travels, but when making at most one intercontinental travel, you will have enough money and time for an additional holiday in Germany. When you would make two intercontinental travels, no time for traveling to Cochem would be left.*

Argumentation can be used to make an informed decision in this scenario: there are three arguments a, b and c for the three respective destinations. a and b attack each other, whereas $\{a, b\}$ attack c. We have represented this as an ADF *consisting of three arguments a, b and c with their respective* acceptance conditions *C_a, C_b and C_c. This results in the* ADF *$D_1 = (\{a, b, c\}, L, C)$ with $L = \{(a, b), (b, a), (a, c), (b, c)\}$ and $C_a = \neg b$, $C_b = \neg a$ and $C_c = \neg a \vee \neg b$. D_1 is represent graphically in Figure 3. Informally, the acceptance conditions can be read as "a is accepted if b is not accepted", "b is accepted if a is not accepted" and "c is accepted if a is not accepted or b is not accepted".*

The argumentative formalisation does not tell us, however, how we should adapt our beliefs in view of changing information. For example, suppose that a highly infectious disease breaks out in Cochem. In that case, argumentative semantics do not give information about what can be expected, unless we change the ADF *in view of this information and recalculate the semantic interpretations for this new* ADF*. However, it might be useful to have an indication of what can be expected in the face of dynamic information. For example, is it reasonable to expect we can still make an intercontinental travel when we do not travel to Cochem (i.e. $\neg c \mathrel{|\!\!\sim}^{\vee}_{a} b$)? The derivation of such statements about what can be expected requires the investigation of belief revision and the resulting dynamic conditionals in the setting of formal argumentation.*

To give a formal account of such revision scenarios, We adapt the AGM-postulates for propositional revision to the setting of revision-operators $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathfrak{D}(\mathsf{At})$ of ADFs by propositional formulas as follows:

Definition 5.11. *An operator* $\star$ *is a* bivalent ADF revision operator *(in short,* $\mathsf{ADF}^2_\star$*-operator) for an ADF* $D = (\mathsf{At}, L, C)$ *and a semantics Sem s.t.* $\mathit{Sem}(D) \subseteq \Omega(D)$[5] *iff* $\star$ *satisfies:*

($\mathsf{ADF}^2_\star 1$)	$D \star \psi \mathrel{\vert\!\sim}^{\cap}_{\mathit{Sem}} \psi$
($\mathsf{ADF}^2_\star 2$)	*If* $\mathit{Sem}(D) \cap \mathsf{Mod}(\psi) \neq \emptyset$ *then* $\mathit{Sem}(D \star \psi) = \mathit{Sem}(D) \cap \mathsf{Mod}(\psi)$
($\mathsf{ADF}^2_\star 3$)	*If* ψ *is satisfiable, then* $\mathit{Sem}(D \star \psi) \neq \emptyset$
($\mathsf{ADF}^2_\star 4$)	*If* $\mathit{Sem}(D) = \mathit{Sem}(D')$ *and* $\psi_1 \equiv \psi_2$ *then* $\mathit{Sem}(D \star \psi_1) = \mathit{Sem}(D' \star \psi_2)$
($\mathsf{ADF}^2_\star 5$)	$\mathit{Sem}(D \star \psi) \cap \mathsf{Mod}(\mu) \subseteq \mathit{Sem}(D \star (\psi \wedge \mu))$
($\mathsf{ADF}^2_\star 6$)	*If* $\mathit{Sem}(D \star \psi) \cap \mathsf{Mod}(\mu) \neq \emptyset$*, then* $\mathit{Sem}(D \star \psi) \cap \mathsf{Mod}(\mu) \supseteq \mathit{Sem}(D \star (\psi \wedge \mu))$

Remark 5.12. *Equivalent formulations of* ($\mathsf{ADF}^2_\star 5$) *and* ($\mathsf{ADF}^2_\star 6$) *(that might be more intuitive to some readers) are:*

($\mathsf{ADF}^2_\star 5$)	$D \star \psi \mathrel{\vert\!\sim}^{\cap}_{\mathsf{2mod}} \mu \to \bigvee \mathit{Sem}(D \star (\psi \wedge \mu))$[6]
($\mathsf{ADF}^2_\star 6$)	*If* $\mathit{Sem}(D \star \psi) \cap \mathsf{Mod}(\mu) \neq \emptyset$*, then* $D \star (\psi \wedge \mu) \mathrel{\vert\!\sim}^{\cap}_{\mathsf{2mod}} (\bigvee \mathit{Sem}(D \star \psi) \wedge \mu)$

These postulates are explained as follows. $\mathsf{ADF}^2_\star 1$ requires that any revision is successful, i. e. the formula that induces the revision should follow from the revised ADF. The second postulate $\mathsf{ADF}^2_\star 2$ requires that if some of the Sem-interpretations of the original ADF satisfy the formula inducing the revision, the revised ADF should have as Sem-interpretations exactly the Sem-interpretations of the original ADF that satisfy the formula inducing the revision. The third postulate states that revising by a consistent formula results in a Sem-consistent ADF, i. e. an ADF that admits Sem-interpretations. $\mathsf{ADF}^2_\star 4$ requires syntax independence: revising ADFs with the same Sem-interpretations by equivalent formulas results in Sem-equivalent revised ADFs. Finally, $\mathsf{ADF}^2_\star 5$ and $\mathsf{ADF}^2_\star 6$ are direct adaptations of the *super-* and *sub-expansion postulates*. They require, in the non-trivial case where $D \star \psi \mathrel{\not\vert\!\sim}^{\cap}_{\mathsf{Sem}} \neg\mu$ (i. e. there is at least one Sem-interpretation of $D \star \psi$ that entails μ, or, in other words, $D \star \psi$ is consistent, under Sem, with μ), that the Sem-interpretations of $D \star (\psi \wedge \mu)$ are exactly the Sem-interpretations of $D \star \psi$ that satisfy μ.

As is usual in work on belief revision, a semantic characterisation in terms of plausibility-orders over interpretations is given. In more detail, we can semantically characterise revision of an ADF D with a formula ϕ in terms of total preorders

[5] The postulates ($\mathsf{ADF}^2_\star 1$)-($\mathsf{ADF}^2_\star 6$) can easily be generalised to a three-valued semantics by substituting $\mathsf{Sem}(D)$ by $\bigcup_{v \in \mathsf{Sem}(D)} [v]^2$. Since we define three-valued revisions below and for reasons of simplicity, we chose to restrict ourselves here to two-valued semantics.

over two-valued interpretations, in analogue to propositional revision. In order to do so, we consider mappings of the type $\mathfrak{D}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$, i. e. functions mapping every ADF D to a total preorder $\preceq_D$ over possible worlds. We first modify Definition 3.7 of an assignment of preorders to be faithful w.r.t. an ADF D and a semantics Sem:

Definition 5.13. *Given a semantics Sem s.t. $\mathsf{Sem}(D) \subseteq \Omega(\mathsf{At})$ for every $D \in \mathsf{At}$, a function $f : D \mapsto \preceq_D$ assigning*[7] *a total preorder $\preceq_D$ over $\Omega(\mathsf{At})$ to every ADF $D \in \mathfrak{D}(\mathsf{At})$ is faithful w.r.t. the semantics Sem iff:*

1. *For every $D \in \mathfrak{D}(\mathsf{At})$, if $\omega, \omega' \in \mathsf{Sem}(D)$, then $\omega \preceq_D \omega'$;*
2. *For every $D \in \mathfrak{D}(\mathsf{At})$, if $\omega \in \mathsf{Sem}(D)$ and $\omega' \notin \mathsf{Sem}(D)$, then $\omega \prec_D \omega'$;*
3. *For every $D, D' \in \mathfrak{D}(\mathsf{At})$, if $\mathsf{Sem}(D) = \mathsf{Sem}(D')$ then $\preceq_D = \preceq_{D'}$.*

The intuition behind a faithful preorder for D (w.r.t. a two-valued semantics Sem) is that the beliefs justified on the basis of an ADF D can be represented as the formulas entailed by all interpretations in $\mathsf{Sem}(D)$ (which is in complete accordance with taking as beliefs all ϕ s.t. $D \mathrel{|\!\sim}^{\cap}_{\mathsf{Sem}} \phi$). A faithful preorder then represents the relative plausibility of formulas (or equivalently, possible worlds) given the ADF D. Therefore, the interpretations sanctioned by D are on the lowermost level, and other interpretations are ranked according to their plausibility by $\preceq_D$.

Example 19. *We illustrate the above definitions by looking at the Dalal-revision operator [17], adapted here to our setting. We first define the symmetric distance function between two possible worlds $\omega, \omega' \in \Omega(\mathsf{At})$ as: $\omega \triangle \omega' = |s \in \mathsf{At} \mid \omega(s) \neq \omega'(s)|$. We can then define $\preceq^{\triangle}_D$ over $\Omega(\mathsf{At})$ by setting*

$$\kappa_{\mathtt{dl}}(\omega) = \min\{\omega' \triangle \omega \mid \omega' \in \mathsf{2mod}(D)\}$$

for any $\omega \in \Omega(\mathsf{At})$ and letting $\omega_1 \preceq^{\triangle}_D \omega_2$ iff $\kappa_{\mathtt{dl}}(\omega_1) \leq \kappa_{\mathtt{dl}}(\omega_2)$.

For the ADF of Example 1, we then obtain the following ranking:

ω	$\kappa_{\mathtt{dl}}$	ω	$\kappa_{\mathtt{dl}}$	ω	$\kappa_{\mathtt{dl}}$	ω	$\kappa_{\mathtt{dl}}$
abc	1	$ab\overline{c}$	2	$a\overline{b}c$	0	$a\overline{b}\overline{c}$	1
$\overline{a}bc$	0	$\overline{a}b\overline{c}$	1	$\overline{a}\overline{b}c$	1	$\overline{a}\overline{b}\overline{c}$	2

[7] Recall that $\Omega(\mathsf{At})$ is the set of all (two-valued) interpretations for S.

We can now semantically characterise revision of an ADF D (under the two-valued semantics Sem) by a formula $\psi \in \mathcal{L}(\mathsf{At})$ as the ADF $D \star \psi$ s.t. :

$$\mathsf{Sem}(D \star \psi) = \min_{\preceq_D}(\mathsf{Mod}(\psi)) \tag{2}$$

Example 20. *Looking again at Example 19, we can use Equation 2 to obtain a revision operator $\star_{\mathtt{dl}}$, which we illustrate by revising D with $\neg c$ based on the preorder $\kappa_{\mathtt{dl}}$ which has as two-valued models:* $\mathsf{2mod}(D \star_{\mathtt{dl}} \neg c) = \{a\overline{b}\overline{c}, \overline{a}b\overline{c}\}$*.*

As we will see below, this revision satisfies all $\mathsf{ADF}^2_\star$-postulates.

Notice firstly that strictly speaking the revision above does not determine a unique ADF. However, it does determine a unique ADF *up to semantical equivalence.* Indeed, in view of Postulate $\mathsf{ADF}^2_\star 4$, we are justified in thus restricting our attention, since the result of the revision of two ADFs D_1 and D_2 with the same Sem-interpretations will result in two ADFs $D_1 \star \phi$ and $D_2 \star \phi$ with the same Sem-interpretations. Secondly, notice that the revision operator defined above is a purely semantical characterisation of revision of ADFs, i. e. the revision of an ADF D by a formula ψ is identified with a set of models. Below we will describe one strategy for obtaining a specific ADF on the basis of the set of two-valued models of an ADF.

In [36], it is shown that the semantic characterisation outlined above is sound and complete:

Corollary 5.14. *Given a finite set of atoms* At*, an operator $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathcal{L}(\mathsf{At})$ is an $\mathsf{ADF}^2_\star$-operator for two-valued model semantics* $\mathsf{2mod}$ *iff there exists a function $f : \mathfrak{D}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ that is faithful w.r.t.* $\mathsf{2mod}$ *s.t.:*

$$\mathsf{2mod}(D \star \psi) = \min_{\preceq_D}(\mathsf{Mod}(\psi))$$

We now move to revision under the stable semantics, where the semantic characterisation is more complicated. In more detail, not every set of two-valued interpretations is realisable under the stable semantics, which means that there might not exist an ADF that has exactly this set of two-valued interpretations as stable models. Indeed, the problem of realisability has been studied in depth by [60]. To characterise revision under stable semantics, we need to ensure realisability of the corresponding faithful mappings. The basic idea is that every "layer" is a $\leq_\top$-antichain. This ensures that every $\preceq_D$-minimal set of two-valued interpretations is realisable under the stable semantics [60]. For example, it is shown there that a set of two-valued interpretations is realisable under the stable semantics if and only if it forms an anti-chain under $\leq_t$, i.e. every two interpretations in the set are $\leq_t$-incomparable. The need for an additional requirement on faithful orderings is shown by the following example

Example 21. *Consider the ADF D from Example 1 and consider $\preceq$ defined as:*

$$\overline{a}bc, a\overline{b}c \prec abc, \overline{a}b\overline{c}, \overline{a}\overline{b}c, a\overline{b}\overline{c} \prec \ldots.$$

Notice that $\preceq$ is faithful w.r.t. stable*. If we revise by $ab \vee \neg c$ by selecting the $\preceq$-minimal models satisfying $ab \vee \neg c$, we obtain* $\mathsf{stable}(D \star (ab \vee \neg c)) = \{abc, \overline{a}b\overline{c}, a\overline{b}\overline{c}\}$. *However, there exists no ADF $(D \star (ab \vee \neg c) \in \mathfrak{D}(\{a, b, c\})$ with $\{abc, \overline{a}b\overline{c}, a\overline{b}\overline{c}\}$ as stable models, since, $\overline{a}b\overline{c} <_\top abc$ contradicts* $\mathsf{stable}(D \star (ab \vee \neg c))$ *forming an $<_\top$-antichain (which we know in view of the results of [60]).*

This problematic behaviour can be avoided by requiring additionally that every layer of a faithful mapping is an $\leq_\top$-antichain:

Definition 5.15. *Given a semantics Sem s.t. $\mathsf{Sem}(D) \subseteq \Omega(\mathsf{At})$ for every $D \in \mathfrak{D}(\mathsf{At})$, a function $f : D \mapsto \preceq_D$ assigning a total preorder $\preceq_D$ over $\Omega(\mathsf{At})$ to every ADF $D \in \mathfrak{D}(\mathsf{At})$ is a* $\top$-modular faithful assignment *w.r.t. the semantics Sem iff:*

1. *if $\omega_1 \preceq_D \omega_2$ and $\omega_2 \preceq_D \omega_1$ then $\omega_1 \not<_\top \omega_2$ and $\omega_2 \not<_\top \omega_1$;*
2. *For every $D \in \mathfrak{D}(\mathsf{At})$, if $\omega, \omega' \in \mathsf{Sem}(D)$ then $\omega' \preceq_D \omega$;*
3. *for every $D \in \mathfrak{D}(\mathsf{At})$, if $\omega \in \mathsf{Sem}(D)$ and $\omega' \notin \mathsf{Sem}(D)$ then $\omega \prec_D \omega'$;*
4. *for every $D, D' \in \mathfrak{D}(\mathsf{At})$, if $\mathsf{Sem}(D) = \mathsf{Sem}(D')$ then $\preceq_D = \preceq_{\mathsf{Sem}(D')}$ for any ADF $D' = (\mathsf{At}, L', C')$.*

Thus, the above definition extends faithful mappings with the requirement that every layer is $\leq_t$-modular.

Example 22. *Consider again the preorder $\preceq$ from Example 21. We can turn this into a T-modular faithful mapping $\preceq'$ as follows (among many other possibilities):*

$$\overline{a}bc, a\overline{b}c \prec' abc \prec' \overline{a}b\overline{c}, \overline{a}\overline{b}c, a\overline{b}\overline{c} \prec' \ldots.$$

Revising D by $ab \vee \neg c$ now results in $\mathsf{stable}(D \star (ab \vee \neg c)) = \{abc\}$. *By the results of [60], $\{abc\}$ is realisable under stable semantics. This illustrates the usefulness of $\top$-modular faithful mappings, as now any selection is ensured to be realisable under stable semantics. This is further ilustrated by the following propositions.*

Theorem 5.16. *An operator $\star : \mathfrak{D}(\mathsf{At}) \times \mathcal{L}(\mathsf{At}) \to \mathcal{L}(\mathsf{At})$ is a revision operator $\star$ for stable semantics iff there exists a function $f : \mathfrak{D}(\mathsf{At}) \to \wp(\Omega(\mathsf{At}) \times \Omega(\mathsf{At}))$ that is $\top$-modular faithful w.r.t.* stable *s.t.:*

$$\mathsf{stable}(D \star \psi) = \min_{\preceq_D}(\mathsf{Mod}(\psi)) \tag{3}$$

5.2.2 Dynamic Conditionals

On the basis of a revision operator, one can define conditional inference based on the *Ramsey-test*. In more detail, we can now stipulate that the conditional $(\psi|\phi)$ follows from the ADF D (relative to a revision operator $\star$ for some semantics Sem), in symbols $D \mathrel{|\!\!\sim}^{\mathsf{Sem}}_{\star} (\psi|\phi)$, iff ψ is in all Sem-models of $D \star \phi$. More informally, the conditional 'if ϕ then usually ψ' is true in the argumentative context D if and only if ψ is true according to all argumentative positions that can be rationally taken in the argumentative context resulting from D revised by ϕ.

We first notice that, given a $\mathsf{ADF}^2_\star$-operator $\star$ and an ADF D, where $\star$ is based on the total preorder $f^\star(D) = \preceq_D$, see Theorem 5.14, a dynamical conditional consequence relation $D \mathrel{|\!\!\sim}^{\mathsf{Sem}}_{\star}$ can be equivalently represented as conditional inference relation induced by the total preorder $\preceq_D$ over Ω. Given a $\mathsf{ADF}^2_\star$-operator satisfying ($\mathsf{ADF}^2_\star 1$)-($\mathsf{ADF}^2_\star 6$), we denote by $f^\star(D)$ the total preorder over Ω induced by $\star$ and D as in Theorem 5.14.[8]

Proposition 5.17. *Given a semantics $\mathit{Sem} \in \{\mathit{2val}, \mathsf{stable}\}$, an ADF D and a $\mathsf{ADF}^2_\star$-operator $\star$ satisfying ($\mathsf{ADF}^2_\star$1)-($\mathsf{ADF}^2_\star$6), $D \mathrel{|\!\!\sim}^{\mathit{Sem}}_{\star} (\psi|\phi)$ iff $\phi \mathrel{|\!\!\sim}_{f^\star(D)} \psi$.*

Thus,conditional inference based on ADFs w.r.t. two-valued semantics is a special case of preferential inference. This stands in contrast with dynamic conditionals based on three-valued semantics, for which an extension to a three-valued logic (such as Kleene's logic) is necessary. For more details, we refer to [36].

Example 23. *Continuing with Example 20, we see that $D \mathrel{|\!\!\sim}^{\mathit{Sem}}_{\star_{\mathtt{dl}}} (a|\neg b)$ in view of $\mathsf{2mod}(D \star_{\mathtt{dl}} \neg c) = \{a\overline{b}\overline{c}, \overline{a}b\overline{c}\}$, i.e. if Cochem is not a viable travel option anymore, we will still go to Addis Aba if we don't go to Boston.*

5.3 Structured Argumentation

In structured argumentation, arguments are not considered as abstract, atomic entities but are kind of rules, consisting of premises and conclusions, making the flow of reasoning more transparent. Since we focus on conditionals in argumentation here, we recall prominent approaches which make use of defeasible rules which are particularly well aligned to conditionals.

[8] Notice that the semantics Sem relative to which a $\mathsf{ADF}^2_\star$-operator is defined are implicitly taken into account in $f^\star(D)$, in the sense that the realisability of this semantics will be taken into account in the additional conditions on the total preorder.

5.3.1 Defeasible Logic Programming and Ranking Functions

*Defeasible Logic Programming (*DeLP*)* [25] combines logic programming with defeasible argumentation. DeLP works in a highly dialectical way, allowing series of attacks and counterattacks to finally mark those statements as *warranted* for which all attackers could be invalidated. Attacks in DeLP are identified via logical contradictions, but the defeat relation needs a preference relation that originally was based on a notion of specificity. The paper [42] makes use of ranking functions [66] and more specific information from System Z [55] to define preference (and hence defeat) between arguments. To this aim, the authors introduce the notions of examples and counterexamples of arguments via possible worlds which are evaluated on the base of ranking functions. The basic idea here is that arguments are as convincing and successful as their most plausible examples, and arguments with more plausible examples should prevail. We recall the basics of this approach from [42], where the strict parts of defeasible logic programs are restricted to be facts.

Let $\mathcal{L}$ be a finitely generated propositional language with atoms $a, b, c, \ldots$, and with formulas $A, B, C, \ldots$, and let Ω denote the set of possible worlds over $\mathcal{L}$. A *defeasible logic program (de.l.p.)* $\mathcal{P} = (\Phi, \Delta)$ consists of a set Φ of facts[9] and a set Δ of defeasible rules which are written as conditionals $\delta = (L|B_1 \ldots B_n)$ with literals $L, B_1, \ldots, B_n$. In accordance with the notions in logic programming, we call L the *head* of the conditional ($L = \mathit{head}(\delta)$) and $B_1 \ldots B_n$ its *body*. Notice that the syntax of rules in DeLP is a special case of that of conditional logics, as the heads consist of single literals and the bodies consist of conjunctions of literals, whereas in conditional logic, any propositional formula is allowed in both the antecedent and the consequent. A literal L can be *defeasibly derived* from $\Delta' \subseteq \Delta$, $\Delta' \mathrel{|\!\!\sim} L$, iff there exists a finite sequence $L_1, \ldots, L_n = L$ of ground literals, such that each L_i is either a fact in Π or there exists a rule in $\Pi \cup \Delta'$ with head L_i and body $\{B_1, \ldots, B_m\}$, and every literal B_j in the body is such that $B_j \in \{L_k\}_{k<i}$. $\Phi \cup \Delta'$ is called *contradictory* iff there is a literal L such that both L and $\overline{L}$ have defeasible derivations from $\Phi \cup \Delta'$. For any *de.l.p.* $\mathcal{P}$, we will presuppose that Φ is non-contradictory.

Given a *de.l.p.* $\mathcal{P} = (\Phi, \Delta)$ and a literal L, $\mathcal{A}$ is an argument for L, denoted $\langle \mathcal{A}, L \rangle$, if $\mathcal{A}$ is a minimal set of defeasible rules in Δ such that there exists a defeasible derivation of L from $\Phi \cup \mathcal{A}$, and $\Phi \cup \mathcal{A}$ is non-contradictory.

An argument $\langle \mathcal{B}, Q \rangle$ is a sub-argument of $\langle \mathcal{A}, L \rangle$ if $\mathcal{B}$ is subset of $\mathcal{A}$. Argument $\langle \mathcal{A}_1, L_1 \rangle$ *attacks*, or *counterargues* another $\langle \mathcal{A}_2, L_2 \rangle$ at a literal L if there exists a sub-argument of $\langle \mathcal{A}_2, L_2 \rangle$, $\langle \mathcal{A}, L \rangle$, i. e., $\mathcal{A} \subseteq \mathcal{A}_2$, such that there exists a literal L' verifying both $\Phi \cup \{L, L_1\} \mathrel{|\!\!\sim} L'$ and $\Phi \cup \{L, L_1\} \mathrel{|\!\!\sim} \overline{L'}$. Note that an argument $\langle \emptyset, L \rangle$ with $L \in \Pi$ can not be attacked since all arguments have to be consistent with

[9] Note that in general, the strict part of a *de.l.p.* also may contain strict rules.

Φ. Finally, another crucial notion involving consistency in DELP is the notion of concordance. A set of arguments $\mathcal{A}_i, 1 \leq i \leq m$, of a defeasible logic program (Φ, Δ) is called *concordant* iff $\Phi \cup \bigcup_{i=1}^{n} \mathcal{A}_i$ is non-contradictory.

Example 24. *We consider the propositional variables b bird, p penguin, c chicken, s is_scared, f flies, w has_wings, and the set of conditionals:* $\Delta = \{\delta_1 = (b|c), \delta_2 = (b|p), \delta_3 = (f|b), \delta_4 = (\overline{f}|p), \delta_5 = (\overline{f}|c), \delta_6 = (f|cs), \delta_7 = (w|b)\}$*. For a de.l.p., this set of conditionals can be instantiated with various facts. For example, consider the defeasible logic program* $\mathcal{P}_1 = (\{cs\}, \Delta)$*. Then the following arguments can be built supporting* f *resp.* $\overline{f}$*:*

$$\begin{array}{ll} \langle \mathcal{A}_1, f \rangle, & \mathcal{A}_1 = \{(b|c), (f|b)\}; \\ \langle \mathcal{A}_2, \overline{f} \rangle, & \mathcal{A}_2 = \{(\overline{f}|c)\}; \\ \langle \mathcal{A}_3, f \rangle, & \mathcal{A}_3 = \{(f|cs)\}. \end{array}$$

Clearly, $\langle \mathcal{A}_2, \overline{f} \rangle$ *attacks* $\langle \mathcal{A}_1, f \rangle$*, and* $\langle \mathcal{A}_3, f \rangle$ *attacks* $\langle \mathcal{A}_2, \overline{f} \rangle$*. Note that* $\{\langle \mathcal{A}_1, f \rangle, \langle \mathcal{A}_3, f \rangle\}$ *is concordant, while* $\{\langle \mathcal{A}_1, f \rangle, \langle \mathcal{A}_2, \overline{f} \rangle, \langle \mathcal{A}_3, f \rangle\}$ *is not.*

As usual in argumentation theory, an attacked argument may not be lost, but can be found to be stronger than its attacker(s). DELP makes use of a preference relation to compare arguments, and in the end, the crucial question in DELP is whether an argument is *warranted.* For the moment, we leave the exact instantiation of the preference relation open because the procedure to ensure warrancy in DELP is the same for any suitable preference relation.

If $\langle \mathcal{A}_1, L_1 \rangle$ and $\langle \mathcal{A}_2, L_2 \rangle$ are two arguments $\langle \mathcal{A}_1, L_1 \rangle$ is a *proper defeater* for $\langle \mathcal{A}_2, L_2 \rangle$ at literal L iff there exists a sub-argument of $\langle \mathcal{A}_2, L_2 \rangle$, $\langle \mathcal{A}, L \rangle$ such that $\langle \mathcal{A}_1, L_1 \rangle$ counterargues $\langle \mathcal{A}_2, L_2 \rangle$ at L and $\langle \mathcal{A}_1, L_1 \rangle$ is strictly preferred over $\langle \mathcal{A}, L \rangle$. Alternatively, $\langle \mathcal{A}_1, L_1 \rangle$ is a *blocking defeater* for $\langle \mathcal{A}_2, L_2 \rangle$ at literal L iff there exists a sub-argument of $\langle \mathcal{A}_2, L_2 \rangle$, $\langle \mathcal{A}, L \rangle$ such that $\langle \mathcal{A}_1, L_1 \rangle$ counterargues $\langle \mathcal{A}_2, L_2 \rangle$ at L and neither $\langle \mathcal{A}_1, L_1 \rangle$ is strictly preferred over $\langle \mathcal{A}, L \rangle$ nor is $\langle \mathcal{A}, L \rangle$ preferred over $\langle \mathcal{A}_1, L_1 \rangle$. If $\langle \mathcal{A}_1, L_1 \rangle$ is either a proper or a blocking defeater of $\langle \mathcal{A}_2, L_2 \rangle$, it is said to be a *defeater* of the latter.

In the warrancy procedure, arguments $\mathcal{A}$ are evaluated in so-called dialectical trees where the root of such a tree is the argument to be evaluated. The paths of the tree consist of (finite) acceptable argumentation lines $[\mathcal{A} = \langle \mathcal{A}_0, L_0 \rangle, \langle \mathcal{A}_1, L_1 \rangle, \langle \mathcal{A}_2, L_2 \rangle, \cdots]$ where each node is a defeater of its parent node, and acceptability of the argument lines is specified by further constraints. Here, it is presupposed that both the sets of *supporting* arguments $[\langle \mathcal{A}_0, L_0 \rangle, \langle \mathcal{A}_2, L_2 \rangle, \langle \mathcal{A}_4, L_4 \rangle, \cdots]$ and *interfering* arguments $[\langle \mathcal{A}_1, L_1 \rangle, \langle \mathcal{A}_3, L_3 \rangle, \langle \mathcal{A}_5, L_5 \rangle, \cdots]$ are concordant. Finally, the nodes are marked U (undefeated) or D (defeated), where a node is marked U iff every child is marked D; in particular, leaves are marked U.

As a novelty, in [42], preference between arguments in DeLP was given an example-based semantics.

Definition 5.18 (Examples, counterexamples)**.** *Let $\mathcal{P} = (\Phi, \Delta)$ be a defeasible logic program. Let $\omega \in \Omega$ be a possible world, and let $\langle \mathcal{A}, L\rangle$ be an argument in $\mathcal{P}$.*

ω is an example *for $\langle \mathcal{A}, L\rangle$ iff ω satisfies all facts, $\omega \models \Phi$, and ω verifies all rules in $\mathcal{A}$. ω is a* counterexample *to $\langle \mathcal{A}, L\rangle$ iff $\omega \models \Phi$ and there is at least one rule in $\mathcal{A}$ that is falsified by ω. ω is a* supported counterexample *to $\langle \mathcal{A}, L\rangle$ iff ω is a counterexample to $\langle \mathcal{A}, L\rangle$ and there is an argument $\langle \mathcal{A}', L'\rangle$ such that ω is an example of $\langle \mathcal{A}', L'\rangle$.*

The set of examples of an argument $\langle \mathcal{A}, L\rangle$ is denoted by $\langle \mathcal{A}, L\rangle^+$, the set of counterexamples by $\langle \mathcal{A}, L\rangle^-$.

From the definition, it is immediately clear that $\langle \mathcal{A}, L\rangle^+ = Mod(\Phi \wedge \bigwedge_{\delta\in\mathcal{A}} head(\delta))$, and $\langle \mathcal{A}, L\rangle^- = Mod(\Phi \wedge \bigvee_{\delta\in\mathcal{A}} \overline{head(\delta)})$. By the definition of arguments, it is ensured that every argument has examples.

Example 25. *For the arguments $\langle \mathcal{A}_1, f\rangle, \langle \mathcal{A}_2, \overline{f}\rangle, \langle \mathcal{A}_3, f\rangle$ stated in example 24, examples and counterexamples are given as follows:*

$$\begin{array}{ll} \langle \mathcal{A}_1, f\rangle^+ = Mod(csbf) & \langle \mathcal{A}_1, f\rangle^- = Mod(cs(\overline{b} \vee \overline{f})) \\ \langle \mathcal{A}_2, \overline{f}\rangle^+ = Mod(cs\overline{f}) & \langle \mathcal{A}_2, \overline{f}\rangle^- = Mod(csf) \\ \langle \mathcal{A}_3, f\rangle^+ = Mod(csf) & \langle \mathcal{A}_3, f\rangle^- = Mod(cs\overline{f}) \end{array}$$

Hence, $\omega_1 = csb\overline{p}fw$ is an example of $\langle \mathcal{A}_1, f\rangle$ and $\langle \mathcal{A}_3, f\rangle$ and a counterexample to $\langle \mathcal{A}_2, \overline{f}\rangle$. Reciprocally, $\omega_2 = csb\overline{p}\overline{f}w$ is an example of $\langle \mathcal{A}_2, \overline{f}\rangle$, and a counterexample to $\langle \mathcal{A}_1, f\rangle$ and $\langle \mathcal{A}_3, f\rangle$.

Attacks can be characterized in terms of examples, as the next proposition shows.

Proposition 5.19. *Let $\langle \mathcal{A}_1, L_1\rangle, \langle \mathcal{A}_2, L_2\rangle$ be two arguments. If $\langle \mathcal{A}_1, L_1\rangle$ attacks $\langle \mathcal{A}_2, L_2\rangle$, then all examples of $\langle \mathcal{A}_1, L_1\rangle$ are (supported) counterexamples to $\langle \mathcal{A}_2, L_2\rangle$, i.e. $\langle \mathcal{A}_1, L_1\rangle^+ \subseteq \langle \mathcal{A}_2, L_2\rangle^-$. Conversely, if all examples of $\langle \mathcal{A}_1, L_1\rangle$ are counterexamples to $\langle \mathcal{A}_2, L_2\rangle$, then there is a sub-argument of $\langle \mathcal{A}_1, L_1\rangle$ that attacks $\langle \mathcal{A}_2, L_2\rangle$.*

Moreover, examples are also helpful to check the crucial notion of concordance in argumentation lines.

Proposition 5.20. *A set of arguments $\langle \mathcal{A}_i, L_i\rangle, 1 \leq i \leq m$, is concordant iff they have common examples, i.e. iff $\bigcap_{1\leq i\leq m} \langle \mathcal{A}_i, L_i\rangle^+ \neq \emptyset$.*

By bringing ranking functions now into the play, plausibility degrees of arguments can be defined. Arguments are assumed to be as plausible as their most plausible examples, and the plausibilities of their counterexamples represent the degree to which they can be challenged. This makes comparisons between arguments easy.

Definition 5.21 (κ-values of arguments, κ-preference)**.** *Let κ be an ordinal conditional function on Ω, let $\langle \mathcal{A}, L\rangle$ be an argument. Then $\kappa^+(\langle \mathcal{A}, L\rangle) = \min\{\kappa(\omega) \mid \omega \in \langle \mathcal{A}, L\rangle^+\}$, and $\kappa^-(\langle \mathcal{A}, L\rangle) = \min\{\kappa(\omega) \mid \omega \in \langle \mathcal{A}, L\rangle^-\}$. Let $\langle \mathcal{A}_1, L_1\rangle$, $\langle \mathcal{A}_2, L_2\rangle$ be two arguments. Then $\langle \mathcal{A}_1, L_1\rangle \succeq^\kappa \langle \mathcal{A}_2, L_2\rangle$ iff $\kappa^+(\langle \mathcal{A}_1, L_1\rangle) \leq \kappa^+(\langle \mathcal{A}_2, L_2\rangle)$.*

From the remarks above, it is immediately clear that $\kappa^+(\langle \mathcal{A}, L\rangle) = \kappa(\Phi \wedge \bigwedge_{\delta\in\mathcal{A}} head(\delta))$ and $\kappa^-(\langle \mathcal{A}, L\rangle) = \kappa(\Phi \wedge \bigvee_{\delta\in\mathcal{A}} \overline{head(\delta)})$.

κ-preference yields a declarative criterion for warrant:

Proposition 5.22. *Let $\langle \mathcal{A}, L\rangle$ be an argument. If*

$$\kappa^+(\langle \mathcal{A}, L\rangle) < \kappa^-(\langle \mathcal{A}, L\rangle)$$

then $\langle \mathcal{A}, L\rangle$ is undefeated and hence warranted.

Of course, when we use a ranking function κ to assess the plausibility of arguments built over a *de.l.p.* $\mathcal{P}$, we expect κ to be a model of Δ. To find such a proper model, we may make use of the distinguished system Z approach [31] as a particularly well-behaved ranking model.

Example 26. *We apply system Z to Δ from $\mathcal{P}_1$ in Example 24. Here, the tolerance partitioning used by system Z is $\Delta_0 = \{\delta_3, \delta_7\}, \Delta_1 = \{\delta_1, \delta_2, \delta_4, \delta_5\}, \Delta_2 = \{\delta_6\}$. We compute the κ_z-values of the arguments in Example 25 as follows:*

$$\begin{array}{ll}
\kappa_z^+(\langle \mathcal{A}_1, f\rangle) = \kappa_z(csbf) = 2 & \kappa_z^-(\langle \mathcal{A}_1, f\rangle) = \kappa_z(cs(\overline{b} \vee \overline{f})) = 2 \\
\kappa_z^+(\langle \mathcal{A}_2, \overline{f}\rangle) = \kappa_z(cs\overline{f}) \;\; = 3 & \kappa_z^-(\langle \mathcal{A}_2, \overline{f}\rangle) = \kappa_z(csf) \qquad\;\; = 2 \\
\kappa_z^+(\langle \mathcal{A}_3, f\rangle) = \kappa_z(csf) \;\; = 2 & \kappa_z^-(\langle \mathcal{A}_3, f\rangle) = \kappa_z(cs\overline{f}) \qquad\;\; = 3
\end{array}$$

From Proposition 5.22, we may conclude immediately that $\langle \mathcal{A}_3, f\rangle$ is a warrant for the literal f.

Let us now consider the defeasible logic program $\mathcal{P}_2 = (\{p\}, \Delta)$. In system Z, we have $\kappa_z(pw) = \kappa_z(p\overline{w}) = 1$, so, the status of the query w can not be determined by system Z. This means that it cannot be proved in system Z if penguins have wings. This effect has become known as the drowning effect *(see, e.g. [31]).*

This problem can be solved in our argumentation framework: The only argument that can be built to connect p and w is $\langle\{(b|p), (w|b)\}, w\rangle$, which is not attacked at all, so, in particular, is undefeated. Hence, w can be warranted. Note, however, that Proposition 5.22 would not be helpful here because $\kappa_z^+(\langle\{(b|p), (w|b)\}, w\rangle) = \kappa_z(pbw) = 1 = \kappa_z(p(\overline{b} \vee \overline{w}) = \kappa_z^-(\langle\{(b|p), (w|b)\}, w\rangle)$.

We thus see here that insights from conditional logics can be made useful for argumentation (e.g. by supplying a preference relation as above), and that argumentation can help improve existing conditional logics (e.g. by helping in avoiding the drowning effect as demonstrated above).

Moreover, in [42], the authors also proposed another preference relation between arguments which is based on system Z by, a bit more simply, comparing the Z-values of the conditionals contained in the involved arguments. This allows for a declarative criterion for ensuring warrancy that just considers the (positive) examples of an argument. For further details, we refer to [42].

5.3.2 Pollock's Defeasible Reasoning and Ranking Functions

Pollock developed a theory of defeasible reasoning [58] where arguments consist of a set of premises and a conclusion which are connected by an inference rule, or reason-schema, respectively. In [67], Spohn briefly discussed the basic ideas of Pollock's work and elaborated on possible connections to his own framework of ranking functions.

The core of Pollock's theory is a large set of defeasible inference rules which can be seen as specific proposals for a constructive theory of defeasible reasoning. Arguments have strengths and can be defeated, and Pollock proposed a formal theory of how defeats and strengths interact in an integrated graph with the aim of arriving at warranted beliefs. Doxastic states are seen as huge networks of inferences and justifications, and all reasoning starts with perceptions.

Spohn appreciated the constructive and dynamic (regarding the flow of reasoning) nature of Pollock's theory, but criticizes it to be basically static because no new information (which are restricted to perceptions in Pollock's theory) can be taken into account in a way that makes the flow of change transparent. For each new perception, the whole reasoning machinery has to start again.

According to Spohn, Pollock's theory overlaps with ranking theory insofar as both approaches deal with justified and warranted belief. However, while ranking theory describes declaratively[10] how such beliefs behave, Pollock's defeasible reasoning implements how such beliefs emerge in many procedural ways. According to [59], all norms of rationality have to be procedural. This seems to be the most crucial difference between both approaches.

The problem with theories of defeasible reasoning like Pollock's approach where inference relies on intuitive procedural rules is that there is no independent assessment of the quality of their products, i.e., warranted beliefs. Spohn called it "normative defectiveness".

[10]Spohn called ranking theory a *regulative theory*.

Defeasible logic programming (DeLP), as described in Section 5.3.1, is also mainly procedural in elaborating warranted beliefs, but it relies on basic logic by exploring contradictions and uses declarative notions like a (more or less) abstract preference relation to determine defeats between arguments. However, while it uses logic programming as kind of a base logic, the semantics of warranted beliefs in DeLP cannot be fully captured by answer set semantics [69]. Nevertheless, DeLP appears to be a good compromise between procedural vs. declarative (or computational vs. regulative, as Spohn termed it in [67]) approaches.

5.3.3 Structured Argumentation Based on Axiomatic Conditional Logic

In the paper [6], the authors extend the deduction-based approach to argumentation from [7] by introducing an additional conditional connective $\Rightarrow$ (giving rise to a logical language $\mathcal{L}_c$) and the novel concept of contrariety between arguments (formulas of $\mathcal{L}_c$). Conditional rules in $\mathcal{L}_c$ specified by $\Rightarrow$ are meant to be hypotheses to be used for tentative reasoning, but which can be attacked by contrary rules in an argumentative process.

For implementing conditional reasoning, Besnard et al. make use of the conditional logic MP [16] which is defined beyond Boolean logic by the following axioms and rules of inference $\vdash_c$:

RCEA $\dfrac{\vdash_c \alpha \leftrightarrow \beta}{\vdash_c (\alpha \Rightarrow \gamma) \leftrightarrow (\beta \Rightarrow \gamma)}$

RCEC $\dfrac{\vdash_c \alpha \leftrightarrow \beta}{\vdash_c (\gamma \Rightarrow \alpha) \leftrightarrow (\gamma \Rightarrow \beta)}$

CC $\vdash_c ((\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma)) \rightarrow (\alpha \Rightarrow (\beta \wedge \gamma))$

CM $\vdash_c (\alpha \Rightarrow (\beta \wedge \gamma)) \rightarrow ((\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma))$

CN $\vdash_c (\alpha \Rightarrow \top)$

MP $\vdash_c (\alpha \Rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$

Note that (RCEC) and (MP) are also axioms of Stalnaker's logic **C2** that we described in Section 3.1. Contrariety is then defined on the base of the logic MP and covers two main cases: first, $\alpha \in \mathcal{L}_c$ is *contrary to* β if both formulas are inconsistent in MP, i.e., $\{\alpha, \beta\} \vdash_c \perp$. The second case deals explicitly with rules involving $\Rightarrow$. The basic idea is that a formula $\alpha = \phi \wedge \epsilon \Rightarrow \psi$ should be *in contrariety to* $\beta = \phi \Rightarrow \psi$ because α suggests that additional preconditions must be satisfied for β to hold. For the precise formal definition of contrariety, we refer to the original paper [6]. If α is in contrariety to β, this is denoted by $\alpha \bowtie \beta$. Note that $\bowtie$ is neither symmetric, nor antisymmetric. Contrariety is lifted to sets of formulas by $\alpha \bowtie \Phi$ if there is $\beta \in \mathcal{L}_c$ such that $\Phi \vdash_c \beta$ and $\alpha \bowtie \beta$.

Given a knowledge base $\Delta \subseteq \mathcal{L}_c$, an argument is a pair $\langle \Phi, \alpha \rangle$ where the following conditions hold:

- $\Phi \subseteq \Delta$;
- for all β such that $\Phi \vdash_c \beta$, $\beta \not\bowtie \Phi$;
- $\Phi \vdash_c \alpha$;
- for all $\Phi' \subset \Phi$, $\Phi' \not\vdash_c \alpha$.

Two arguments $\langle \Phi, \alpha \rangle, \langle \Psi, \beta \rangle$ are *quasi-identical* if $\Phi = \Psi$ and $\alpha \equiv_c \beta$, where $\alpha \equiv_c \beta$ means $\alpha \vdash_c \beta$ and $\beta \vdash_c \alpha$.

Attacks between arguments are defined in terms of contrariety. An argument $\langle \Psi, \beta \rangle$ is a *rebuttal* for $\langle \Phi, \alpha \rangle$ if $\beta \bowtie \alpha$, and $\langle \Psi, \beta \rangle$ is a *defeater* for $\langle \Phi, \alpha \rangle$ if $\beta \bowtie \Phi$. Besnard et al. show that rebuttals are subsumed by defeaters so that we can focus on defeaters from now on. However, defeaters can be quite general so that we need additional attributes to characterize most relevant defeaters. First, defeaters should be most specific both in a set-theoretical and logical sense: an argument $\langle \Phi, \alpha \rangle$ is at least as *conservative* than an argument $\langle \Psi, \beta \rangle$ if $\Phi \subseteq \Psi$ and $\beta \vdash_c \alpha$. In the following, an enumeration $\langle \Psi_1, \beta_1 \rangle, \langle \Psi_2, \beta_2 \rangle \ldots$ of all maximally conservative defeaters for $\langle \Phi, \alpha \rangle$ is assumed to be fixed for each argument. $\langle \Psi_i, \beta_i \rangle$ is a *pertinent defeater* for $\langle \Phi, \alpha \rangle$ if for each $j < i$, $\langle \Psi_i, \beta_i \rangle$ and $\langle \Psi_j, \beta_j \rangle$ are not quasi-identical.

Finally, pertinent defeaters are used to build argumentation trees. An *argumentation tree* for α has an argument for α as its root, and each child node is a pertinent defeater of its parent node; moreover, for each node $\langle \Psi, \beta \rangle$ with ancestor nodes $\langle \Psi'_1, \beta'_1 \rangle, \ldots, \langle \Psi'_n, \beta'_n \rangle$, we have $\Psi \not\subseteq \Psi'_i$ for $i \in \{1, \ldots, n\}$. Argumentation in this conditional logic then may follow the lines of the classical framework in [7].

It is interesting to note that the semantics for conditionals provided by ranking functions [66] which is used to equip DeLP argumentation with an example-based semantics in Section 5.3.1 satisfies the axioms and inference rules of the conditional logic MP (under mild prerequisites). This can easily be verified by observing that RCEA, RCEC, CC, and CM are implied by system P [44] (see also Section 3.1) which inference based on ranking functions is known to satisfy, and MP is a simple arithmetic exercise for ranking functions. CN holds for consistent formulas as all ranks assigned to worlds are finite. An interesting research question would be if ranking functions can also provide a semantics for the approach presented in [6], and how contrariety can be characterized in terms of ranking functions.

5.4 Other approaches

We now shortly discuss some other approaches that can be argued to connect structured argumentation and conditional logics.

Gabbay and d'Avila Garcez [24] ask a methodological question about structured argumentation by giving detailed considerations on the different options for instantiating abstract argumentation frameworks. This paper argues that there is a wide variety of options to do so, and gives several detailed examples of how this can be done. Among others, non-monotonic logics, i.e. consequence relations satisfying reflexivity, cut and cautious monotony, are discussed. In more detail, Gabbay and d'Avila Garcez suggest that nodes in an argumentation graph could represent pairs of sets of non-monotonic conditionals and a conclusion based on these conditionals, and that an argument (Δ, ϕ) attacks an argument (Θ, ψ) if adding ϕ leads to ψ not being derivable anymore in view of Θ. For example (adapting the notation somewhat to our article), $(\{p\}, p)$ attacks $(\{b, b \mid\!\sim'_f p \wedge b \mid\!\sim^{f}_{\neg}\}, f)$ as the knowledge that something is a penguin no longer allows us to derive that it flies according to most non-monotonic logics.

A brand of non-monotonic logics that allows to reason with conditional statements that we have not discussed here are *input/output-logics* [52]. They provide a fine-grained picture of the different ways of reaching a conclusion by forward chaining a selected subset of conditionals, and have been proven especially useful in deontic logics. These logics are given an argumentative characterisation by Van Berkel and Straßer [74]. They do this by defining *deontic argument calculi*, which allow for structured argumentation on the basis of a set of conditionals interpreted as normative statements. The different input/output-logics from the literature are then captured by allowing for different inference rules in the process of argument construction, many of which are quite familiar to the axioms from conditional logics. It is an interesting question whether also the conditional logics discussed above can be represented in a similar way.

6 Further works

Both conditional logic and abstract argumentation (or some extension of it such as ADFs) are logical formalisms for reasoning. In this article, we discussed several ideas on how to combine these formalisms into a single formalism. Our focus was on works where we used the foundational ideas of semantical evaluation from one formalism and applied it in the other. Another general approach for combining arbitrary logics into a joint formalisms is that of *fibring*, see [22, 23]. Given two logics $\mathcal{L}_1$ and $\mathcal{L}_2$, the fibring $\mathcal{L}_{1,2}$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ that combines both syntax and semantics for both base logics in a simple manner. The syntax of $\mathcal{L}_{1,2}$ allows for an arbitrary combination of the syntax of $\mathcal{L}_1$ and $\mathcal{L}_2$, e. g., formulas may contain connectors of both $\mathcal{L}_1$ and $\mathcal{L}_2$ in an arbitrary manner. Informally speaking, if one were to fibre conditional logic and abstract argumentation, a valid formula would be $(aRb|bRc)$ with the intuitive meaning "if b attacks c, then usually a attacks b". The semantics of $\mathcal{L}_{1,2}$ is then a combination of the semantics of both $\mathcal{L}_1$ and $\mathcal{L}_2$ as well. In particular, [23] defines an inference relation $\mid\!\sim_{1,2}$ on $\mathcal{L}_{1,2}$ that is a conservative extension of given inference relations $\mid\!\sim_1$ and $\mid\!\sim_2$ on $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, in the sense, that $\mathcal{L}_{1,2}$ coincides with $\mathcal{L}_1$ and $\mathcal{L}_2$ on the respective syntactical fragments of $\mathcal{L}_{1,2}$. However, properties of this new inference relation $\mid\!\sim_{1,2}$ cannot be derived in a general manner and depend highly on the logics $\mathcal{L}_1$ and $\mathcal{L}_2$ and their inference relations $\mid\!\sim_1$ and $\mid\!\sim_2$, respectively. In essence, fibring logics allows for *joint* reasoning of two different formalisms in one single framework, while most of the work discussed in this article was concerned with an *integrated* approach to reasoning. How exactly a fibred logic using conditional logic and abstract argumentation (or ADFs) behaves, could be an interesting avenue for future work, though.

[77, 78, 79] presents a new semantics for abstract argumentation, which is also rooted in conditional logical terms. In more detail, a ranking interpretation is provided for extensions of arguments instantiated by strict and defeasible rules by using conditional ranking semantics. Thus, Weydert presupposes a conditional knowledge base that is used to construct an argumentation framework.

[10] relates Abstract Dialectical Frameworks to causal reasoning, and, more precisely, to Pearl's causal models [56]. In essence, a causal model describes causal dependencies between *exogeneous* variables (which cannot directly be observed) and *endogenous* variables, which can be observed. A causal model formalises how states of variables are caused by other states of variables and, due to the non-monotonicity of causality, a causal model can thus be interpreted as a specific non-monotonic theory, quite similar to conditional logics. Bochman then shows certain correspondences between semantical notions of ADFs and causal models by modelling acceptance conditions of ADFs as causal rules. A previous study by [8] already revealed similar relationships between assumption-based argumentation by [12] and the causal reasoning approach of [30].

Another contribution of Alexander Bochman [9] proposes a conceptual differentiation between two paradigms of non-monotonic reasoning, which he calls *preferential* and *explanatory*. The conditional logics discussed above fall under the first paradigm, whereas argumentation is an example of a formalism for explanatory non-monotonic reasoning. A number of differences between the two kinds of non-monotonic reasoning are discussed, and a general axiomatic theory for each of these paradigms is given. Even though the general conclusion of [9] agrees with the insights expounded in this overview, we leave a deeper comparison between these works for the future.

Verheij has initiated a line of work [76] that integrates ideas from preferential reasoning into argumentation by means of so-called *case models*. A case model $(C, \geq)$ consists of a set C of logically consistent, mutually incompatible formulas and a total preorder $\geq$ over these cases. Arguments, conceived of as pairs of formulas (ϕ, ψ) representing the premise ϕ and conclusion ψ, are then classified on the basis of a case model using ideas inspired by preferential semantics. For example, an argument (ϕ, ψ) is *presumptively valid*[11] if, among all cases verifying the premise ϕ, there is a $\geq$-maximal case that also verifies ψ.

[3, 4] consider preferential interpretations for abstract argumentation frameworks that are derived from gradual semantics. The latter allow to assign numeric values of argument strength to individual arguments and are therefore similar to ranking functions (and therefore preferential interpretations) for conditional logics. This approach allows to reason over argument acceptance (and arbitrary formulas over arguments) through defeasible rules that can be derived from the preferential interpretations.

[71, 72] introduce *stratified labelings*, a semantical approach to abstract argumentation frameworks, where arguments receive a non-negative natural number that assesses the *controversiality* of arguments and are inspired by ordinal ranking functions from conditional logics. As a matter of fact, [71] show that conditional knowledge bases can be transformed into abstract argumentation frameworks, such that rational stratified labelings of the latter behave similarly as the system Z ranking function of the former.

The behaviour of abstract argumentation in dynamic settings in is studied [63]. In more detail, they ask the question whether the labelling status of arguments is preserved when adding or removing arguments or attacks in an abstract argumentation framework. This conceptually is quite similar to postulates such as (cautious) monotony, where beliefs persist when adding (believed) formulas.

Finally, we notice that some foundational papers on non-monotonic conditionals expand on ideas that are, at least conceptually, related to argumentation. For example, in Lehmann and Magidor's prolific paper on rational closure [46], the authors motivate the preference-comparison between cumulative models using the notions of attack and defence. Geffner and Pearl [28] go even further, giving a full-fledged argumentative proof theory that is sound and complete with respect to their conditional inference method of *conditional entailment*.

[11] [76] uses different notations for different kinds of arguments, e.g. a presumptively valid argument is denoted by $\phi \rightsquigarrow \psi$.

There are further works that only loosely touch on the subject of this article, but still model some aspect of *conditional inference*. For example, [5] introduce *conditional preference-based argumentation frameworks*, which allow the specification of preferences between arguments. In fact, preferences are given conditioned on selected sets of arguments and can differ for different sets. A similar approach for structured argumentation is discussed by [20]. [57] define *conditional labels* for arguments that describe conditions about the acceptance of arguments, given the status of other arguments (similarly as acceptance conditions in ADFs). These can be used in dialogues to enable strategic moves of agents. Another form of *conditional labelings* are presented by [13]. Here, a conditional labeling assigns acceptance status to arguments, under the condition that another set of labelings is assumed to evaluate the argumentation framework rationally. Using conditional labelings, the strict semantical evaluation of classical semantics can be relaxed and conditional labelings models rationality as close as possible, given the circumstances. The work of [13] therefore shares some motivation with the work of [64, 65] that we discussed in Section 5.

7 Summary and Conclusion

In this article, we gave a thorough introduction to the logic of conditionals, and have surveyed work that integrated ideas inspired by conditional logics into formal argumentation. We saw that despite the differences between the two approaches (Section 5), integrating insights from conditional logic into formal argumentation is still useful and results in richer argumentative models (as demonstrated in Section 4 and 5.3), while argumentative models can also improve upon conditional logics (as we saw in Section 5.3). As indicated in several parts of this article, we believe there is still a lot of exciting work to be done in this area, and hope our article will serve as a useful basis for such further investigations.

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